1 Exercise 34.1 Solution - Guessing $2/3$ of the Average

The set of strategies, $(1, 1, 1)$ is a Nash Equilibrium (NE). Deviating to $(2, 1, 1)$ by player 1, for example, raises the average to $4/3$, so $2/3$ of the average is $8/9$. Players two and three would split the prize. To rigerously show this is a NE, we start with a set of strategies (k, k, k), with $k \geq 2$ and show that all players have an optimal deviation to $k - 1$. The reasoning is as follows:

- Suppose the strategies are (k, k, k) with $k \geq 2$. With these strategies, all three players split the prize. WLOG, consider a deviation by player 1 to $(k - 1, k, k)$. Thus:

$$\overline{k} = k - \frac{1}{3}.$$

$$\frac{2}{3}\overline{k} = \frac{2}{3}k - \frac{2}{9} = \theta.$$

  So the player with the choice closest to θ wins.
  The distance between player two and three's choice and θ is:

$$\begin{aligned} D_{23} &= |k - \theta| \\ &= |k - (\frac{2}{3}k - \frac{2}{9})| \\ &= |\frac{1}{3}k + \frac{2}{9}| \end{aligned}$$

  The distance between player one's choice and θ is:

$$\begin{aligned} D_1 &= |k - 1 - \theta| \\ &= |k - 1 - (\frac{2}{3}k - \frac{2}{9})| \\ &= |\frac{1}{3}k - \frac{7}{9}| \end{aligned}$$

  So $D_{23} > D_1 \; \forall \; k \geq 2$, so player one would win if she deviated to $k - 1$. At $(1, 1, 1)$, no further deviations are possible, so we have reached a NE.

But are there other NE in pure strategies? We have to check two more cases.

- Suppose the strategies are (k_1, k_2, k^*) with $k^* > k_1 \geq k_2$. So one player chooses a strictly higher value than the other two. The other two players may or may not play the same value. (Note it's equivalent to have player two playing k^*, and players one and three playing the the lower values). Thus we have:

$$\overline{k} = \frac{1}{3}(k^* + k_1 + k_2).$$

$$\frac{2}{3}\overline{k} = \frac{2}{9}(k^* + k_1 + k_2) = \theta.$$

So the player with the choice closest to θ wins. Clearly if $\theta < k_1$, the player playing k^* would not win so she should lower her selection. Suppose $\theta \in (k_1, k^*)$.
The distance between player three's choice and θ is:

$$\begin{aligned} D_3 &= k^* - \theta \\ &= k^* - \frac{2}{9}(k^* + k_1 + k_2) \\ &= \frac{7}{9}k^* - \frac{2}{9}k_1 - \frac{2}{9}k_2 \end{aligned}$$

The distance between player one's choice and θ is:

$$\begin{aligned} D_1 &= \theta - k_1 \\ &= \frac{2}{9}(k^* + k_1 + k_2) - k_1 \\ &= \frac{2}{9}k^* - \frac{7}{9}k_1 + \frac{2}{9}k_2 \end{aligned}$$

Note we didn't need absolute values because both expressions are positive. So playing k^* is NOT optimal if $D_3 > D_1$. That is if:

$$\begin{aligned} \frac{7}{9}k^* - \frac{2}{9}k_1 - \frac{2}{9}k_2 &> \frac{2}{9}k^* - \frac{7}{9}k_1 + \frac{2}{9}k_2 \\ \frac{5}{9}k^* + \frac{5}{9}k_1 - \frac{4}{9}k_2 &> 0 \\ 5k^* + 5k_1 - 4k_2 &> 0 \end{aligned}$$

The final expression holds because $k^* > k_1 \geq k_2$. Thus playing k^* is not optimal. Thus (k_1, k_2, k^*) is NOT a NE.

The final case to check is the following:

- Suppose the strategies are (k^*, k^*, k) with $k^* > k$. So two players chooses a strictly higher value than the last. Thus we have:

$$\overline{k} = \frac{1}{3}(k^* + k^* + k) = \frac{2}{3}k^* + \frac{1}{3}k.$$

$$\frac{2}{3}\overline{k} = \frac{4}{9}k^* + \frac{2}{9}k = \theta.$$

In this case, players one and two both should deviate to k. One way to show this is to notice that the average of k and k^* is $\frac{1}{2}k^* + \frac{1}{2}k$. Thus,

$$\theta = \frac{4}{9}k^* + \frac{2}{9}k < \frac{1}{2}k^* + \frac{1}{2}k,$$

because $\frac{4}{9} < \frac{1}{2}$ and $\frac{2}{9} < \frac{1}{2}$. So θ is closer to k than it is to k^*. Player three wins. Players one and two have an optimal deviation to playing k. Thus (k^*, k^*, k) is NOT a NE.

So given this argument, the only NE is $(1, 1, 1)$.