Economics 604: Microeconomics
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1 Lecture 1: January 27, 2005

1.1 General Equilibrium Model (GE)

- Primitives: agents (consumers and firms), preferences, technologies and resources.
- The GE model assumes perfect competition (price taking behavior) so no firms have any market power and there are no powerful consumers. Prices must clear in all markets simultaneously.
- An equilibrium consists of a list of prices, one for each market.
- The model is good for evaluating policies (especially macro type issues), however it is only a benchmark and is NOT realistic.
- Very similar to Adam Smith's Invisible Hand theory: it explains how a price system may act to coordinate decentralized activities.
- Outline of course:
  - (1) Existence: Is there an equilibrium? Does there exist a list of prices to clear all markets?
  - (2) Efficiency: Not a very strong concept (Size of the pie, welfare theorems, etc).
  - (3) Uniqueness: Might there exist more than one equilibrium?
  - (4) Stability: Should we expect prices to change dramatically after a small preturbation?
  - (5) Applications: Social Choice, International Trade, Finance.
- The main weakness of the GE model is the price taking assumption and the main strength is it provides an elegant analysis of interactions between markets.
- While game theory dealt with partial equilibriums in markets (and allowed for market power), the GE model keeps it simple and solves in all markets but with no market power. Thus it can be seen as a complement to game theoretic models.

2 Lecture 2: February 1, 2005

2.1 GE Theory - Preview

- The Main themes in GE theory are Walrasian Equilibrium, existence, Uniqueness, and Efficiency.

- Consider an Edgeworth box pure exchange economy with no firms. There are 2 consumers (index i) and 2 goods (index l).

- **Definition:** Endowments:
$$\omega_i = (\omega_{1i}, \omega_{2i}),\ i = 1, 2.$$
Note we index with the good first, then the consumer.

- **Definition:** Demands (consumption):
$$x_i = (x_{1i}, x_{2i}),\ i = 1, 2.$$

- Consumers have preferences over the two goods which are complete, transitive and monotone.

- Example. We will outline an example throughout all of this. Assume utility takes the form:
$$u_1(x_{11}, x_{21}) = x_{11}^{\alpha} x_{21}^{1-\alpha},$$
$$u_2(x_{12}, x_{22}) = x_{12}^{\alpha} x_{22}^{1-\alpha}.$$
Endowments:
$$\omega_1 = (1, 2),$$
$$\omega_2 = (2, 1).$$
Thus, preferences and endowments are a complete description of the economy! It's all we need. See G-2.1.

- **Definition:** An allocation is:
$$x = (x_1, x_2) = ((x_{11}, x_{21}), (x_{12}, x_{22})) \in \Re^4_+.$$

- **Definition:** A feasible allocation satisfies:
$$x_{11} + x_{12} \leq \omega_{11} + \omega_{12} = \overline{\omega}_1.$$
$$x_{21} + x_{22} \leq \omega_{21} + \omega_{22} = \overline{\omega}_2.$$
In our example, $\overline{\omega}_1 = \overline{\omega}_2 = 3$.

- See G-2.2 for a complete edgeworth box for our example. In G-2.3 we overlay preferences in the edgeworth box.

- In general, we assume this is a competitive economy where consumers choose optimal bundles in their budget sets taking prices as given. Market prices must also clear (Supply=Demand) in both markets. An important note here is that the consumer's wealth is endogeneously determined by the market prices since his income comes from selling his endowment in the market. This is a characteristic of GE which you don't see in partial equilibrium models.

- **Definition:** The Budget set of consumer i:

$$B_i(p) = \{x_i = (x_{1i}, x_{2i}) \in p_1x_{1i} + p_2x_{2i} \leq p_1\omega_{1i} + p_2\omega_{2i}\}.$$

  It's the set of all affordable bundles given prices, p. For our example:

$$B_1(p) = \{x_1 = (x_{11}, x_{21}) \in p_1x_{11} + p_2x_{21} \leq p_1 + 2p_2\},$$

$$B_2(p) = \{x_2 = (x_{12}, x_{22}) \in p_1x_{12} + p_2x_{22} \leq 2p_1 + p_2\}.$$

- **Definition:** The Budget line of course is:

$$x_i \ni p_1x_{1i} + p_2x_{2i} = p \cdot \omega_i.$$

  Using the dot product notation on the right.

- As usual $-\frac{p_1}{p_2}$ will be the slope of the budget line.

- **Definition:** Consumer's wealth is thus:

$$w_i = p \cdot \omega_i.$$

- Note that ω_i, the initial allocation is ALWAYS on the budget line. You can always afford your endowment. See G-2.4 and G-2.5. Note we can depict the budget sets on the same box as well.

- In G-2.6 we show that for a given endowment point, the optimal level of demand by consumer 1 is shown at the point x_1.

- **Definition:** A Walrasian Equilibrium is a price vector, p^* and a feasible allocation $x^* = (x_1^*, x_2^*)$ in the Edgeworth box such that:

  - (1) Both consumers are maximizing utility:

$$x_i^* \succeq x_i' \; \forall \; x_i' \in B_i(p^*) \text{ for } i = 1, 2.$$

  - (2) Both markets clear:

$$x_{l1} + x_{l2} = \omega_{l1} + \omega_{l2} \text{ for } l = 1, 2$$

  Note that the second condition is really redudant because at any feasible point in the edgeworth box, markets will clear.

- We may have a disequilibrium in prices, as shown in G-2.7. Here neither market clears as there is excess demand for good 2 and excess supply of good 1.
- In G-2.8 we have a Walrasian equilibrium, both markets clear.
- So how do we find a Walrasian equilibrium in an edgeworth box?
  - (1) For each p, find the consumers demand for each good.
  - (2) Use the market clearing condition to find the prices such that both markets clear.
- Back to our example. Consider the problem of consumer 1:

$$Max\ u_1 = x_{11}^{\alpha} x_{21}^{1-\alpha},$$

s.t.

$$p_1 x_{11} + p_2 x_{21} = \underbrace{p_1 + 2p_2}_{w_1}.$$

The lagrangian:

$$\mathcal{L} = x_{11}^{\alpha} x_{21}^{1-\alpha} + \lambda[p_1 + 2p_2 - p_1 x_{11} - p_2 x_{21}].$$

FOC(x_{11}):

$$\alpha x_{11}^{\alpha-1} x_{21}^{1-\alpha} - \lambda p_1 = 0.$$

FOC(x_{21}):

$$(1-\alpha) x_{11}^{\alpha} x_{21}^{-\alpha} - \lambda p_2 = 0.$$

Divide the first by the second:

$$\frac{\alpha x_{11}^{\alpha-1} x_{21}^{1-\alpha}}{(1-\alpha) x_{11}^{\alpha} x_{21}^{-\alpha}} = \frac{p_1}{p_2}.$$

$$\frac{\alpha}{1-\alpha} \frac{x_{21}}{x_{11}} = \frac{p_1}{p_2}.$$

Rewrite:

$$p_2 x_{21} = \frac{1-\alpha}{\alpha} p_1 x_{11}.$$

Substitute into the budget constraint:

$$p_1 x_{11} + p_2 x_{21} = w_1.$$

$$p_1 x_{11} + \frac{1-\alpha}{\alpha} p_1 x_{11} = w_1.$$

$$\frac{1}{\alpha} p_1 x_{11} = w_1.$$

$$p_1 x_{11} = \alpha w_1.$$

And similarly,

$$p_2 x_{21} = (1-\alpha) w_1.$$

And these are the familiar Cobb-Douglas results for Walrasian demands.

- Solving for x:
$$x_{11} = \frac{\alpha w_1}{p_1} = \frac{\alpha(p_1 + 2p_2)}{p_1},$$
$$x_{21} = \frac{(1-\alpha)w_1}{p_2} = \frac{(1-\alpha)(p_1 + 2p_2)}{p_2}.$$
Thus Walrasian demands are:
$$x_1(p, p\cdot(1,2)) = \left[\left(\frac{\alpha(p_1+2p_2)}{p_1}\right), \left(\frac{(1-\alpha)(p_1+2p_2)}{p_2}\right)\right],$$
$$x_2(p, p\cdot(2,1)) = \left[\left(\frac{\alpha(2p_1+p_2)}{p_1}\right), \left(\frac{(1-\alpha)(2p_1+p_2)}{p_2}\right)\right].$$

- Now we impose the market clearing condition that total demand for each good must equal the sum of the endowments. Thus,
$$\frac{\alpha(p_1+2p_2)}{p_1} + \frac{\alpha(2p_1+p_2)}{p_1} = 3.$$
$$\frac{(1-\alpha)(p_1+2p_2)}{p_2} + \frac{(1-\alpha)(2p_1+p_2)}{p_2} = 3.$$
Solving, we find:
$$\frac{p_1^*}{p_2^*} = \frac{\alpha}{1-\alpha}.$$
So we cannot pin down actual prices, just the ratio. This is the usual result because if you double all the prices, income also doubles so we're left off at the same point. Also, we didn't need both of the last two market clearing conditions to solve for the price ratio, one is enough by Walras Law. If there are N markets and $N-1$ of them are in equilibrium, the other has to be as well.

- Now plug the price ratio into the Walrasian demands to get the equilibrium:
$$x_{11} = x_{21} = 2 - \alpha.$$
$$x_{12} = x_{22} = 1 + \alpha.$$
See G-2.9 for the Edgeworth box of the solution.

- **Definition:** See G-2.10 and G-2.11 for the dipiction of Offer Curves. The offer curve is a function of the price ratio and it shows the optimal demand by a consumer for any given price ratio. A Walrasian demand is at the intersection of the two consumer's offer curves.

- Can an exchange economy have more than one equilibrium, yes. It turns out we need STRICTLY convex preferences to get uniqueness. Convex preferences, as with quasi-linear utility, is not enough. See G-2.12 for the edgeworth box when consumers have quasi-linear preferences.

- Does there always exist a solution? No. Convexity is a sufficient condition for existence (not necessary), but without it, there may not be a solution. If a consumer has non-convex connoisseur prefences, where he attains utility from the larger of the quantities of the two bundles, and he competes with a consumer that has left shoe/right shoe preferences, there will not be any equilibrium.
- If both consumers have non-convex preferences, there may be an equilibrium. These are all extreme solutions, but worth studying for an exam!

3 Lecture 3: February 3, 2005

3.1 Walrasian Equilibrium and Efficiency

Model Setup

- In this section, we are moving towards the concept of a Walrasian equilibrium. 4 things must happen:
  - (1) Pricing taking firms and consumers.
  - (2) Consumers maximize their utility over their budget sets.
  - (3) Firms maximize their profits over their production set.
  - (4) Markets must clear.
- Consumers are indexed by $i = 1, \ldots, I$, firms $j = 1, \ldots, J$, and goods $l = 1, \ldots, L$.
- Consumer preferences are complete and transitive on $\Re^L_+$.
- Endowment of the consumers are:

$$\omega_i = (\omega_{1i}, \omega_{2i}, \ldots, \omega_{Li}) \in \Re^L_+.$$

- The total endowment of all goods:

$$\overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2, \ldots, \overline{\omega}_L),$$

  with,

$$\overline{\omega}_1 = \sum_{i=1}^{I} \omega_{1i}.$$

- Firms have a production set $Y_j \subset \Re^L$ or:

$$y_j = (y_{1j}, y_{2j}, \ldots, y_{Lj}).$$

- Example. In a two good production economy where the two goods are leisure/labor (h) and coconuts (c), the production function may be:

$$c = f(h) = \sqrt{h}.$$

- See G-3.1. Here we show the production set graphically. Inputs are usually denoted by a negative number (hence we usually use the NW quadrant), and outputs are positive. Thus firm's technology can be represented by this production set or simply by the list of goods above (again, negative for inputs, positive for outputs).
- We can also designate the production set notationally:

$$Y = \{(-h, c) | 0 \le h < \infty, 0 \le c \le h^{1/2}\}.$$

- **Definition:** An allocation is:

$$(x, y) = (x_1, \ldots, x_I, y_1, \ldots, y_J),$$

where x_i is the consumption of consumer i and y_j is the production vector of firm j. (x, y) is feasible if for each good l,

$$\underbrace{\sum_i x_{li}}_{Demand} \leq \underbrace{\overline{\omega}_l + \sum_j y_{lj}}_{Supply}.$$

So the total consumption of good l must be less than or equal to the endowment (by consumers) and production (by firms).

- Profit Maximization by the firm. Firm's take prices, (P_h, P_c) as given and solve:

$$Max_{h,c}\;\; P_c c - P_h h,$$

subject to:

$$(-h, c) \in Y \;\; or \;\; c \leq \sqrt{h}.$$

Lagrangian:

$$\mathcal{L} = P_c c - P_h h + \lambda[h^{1/2} - c].$$

FOCs:

$$P_c - \lambda = 0.$$
$$P_h - \frac{1}{2}h^{-1/2}\lambda = 0.$$
$$\sqrt{h} - c = 0.$$

The first two conditions imply:

$$P_c = 2P_h h^{1/2} \Longrightarrow \frac{P_c}{P_h} = 2\sqrt{h}.$$

Plug in the last condition:

$$\frac{P_c}{P_h} = 2c \Longrightarrow c^* = \frac{P_c}{2P_h}.$$

Also,

$$h^* = \left(\frac{P_c}{2P_h}\right)^2.$$

And profits:

$$P_c c - P_h h = P_c\frac{P_c}{2P_h} - P_h\left(\frac{P_c}{2P_h}\right)^2 = \frac{P_c^2}{2P_h} - \frac{P_c^2}{4P_h} = \frac{P_c^2}{4P_h}.$$

- Consumer's have wealth equal to their endowments, but also from shares they own in the firm. Since this is a closed economy, profits must go somewhere, so we assume that the firms are owned by the consumers. Denote, $\theta_{ij} \in [0,1]$, consumer i's share of firm j. Of course,

$$\sum_{i=1}^{I} \theta_{ij} = 1 \ \forall \ j = 1, \ldots, J.$$

Thus the budget set of a typical consumer (1) is denoted:

$$B_1(p) = \{x_1 = (x_{11}, \ldots, x_{L1}) \in \Re_+^L | p \cdot x_1 \leq w_1\},$$

where,

$$w_1 = p \cdot \omega_1 + \sum_{j=1}^{J} \theta_{1j}(p \cdot y_j).$$

- An allocation (x^*, y^*) and a price vector $p = (p_1, \ldots, p_L)$ are a Walrasian equilibrium if:
  - (1) Each firm maximizes profits in Y_j:

$$p \cdot y_j^* \geq p \cdot y_j \ \forall \ y_j \in Y_j.$$

  - (2) Each consumers maximizes utility in $B_i(p)$:

$$x_i^* \succeq_i x_i \ \forall \ x_i \in B_i(p).$$

  - (3) Markets clear for each good:

$$\sum_{i=1}^{I} x_i^* = \overline{\omega} + \sum_{j=1}^{J} y_j^*.$$

Robinson Crusoe Economy

- Assume there is one consumer on an island who operates one firm. There are 2 goods:

$$h = \text{ hours of labor or leisure},$$

$$c = \text{ coconuts}.$$

- Technology, $c = \sqrt{h}$. Utility for Robinson:

$$U(h, c) = h^{1-\beta} c^{\beta}, \ \beta \in (0, 1).$$

Here h is hours of leisure. Endowments:

$$\omega_h = T, \ \ \omega_c = 0.$$

So Robinson choose h hours of leisure leaving $T - h$ hours of labor to be sold to the firm.

- A Walrasian equilibrium is a set of prices for coconuts and labor/leisure, and then consumptions of coconuts and leisure by Robinson, and production of coconuts using labor, again by Robinson!

- Robinson's problem from his "consumer-self":

$$Max_{h,c} \;\; h^{1-\beta}c^{\beta},$$

  such that,

$$(h,c) \in B(p) = \{(h,c) : P_c c + P_h h \leq w\}.$$

  Where Robinson's total wealth is:

$$w = P_h T + \pi(P_c, P_h).$$

  So this is important. We can think of Robinson initially selling his total time endowment to the firm at a price P_h, netting him income of $P_h T$. He then buys leisure from the firm as shown in his budget constraint. Note that the other part of Robinson's wealth comes from the firm's profits. This is NOT internalized! From Robinson's "consumer-self" he treats these profits as external, even though his decision about how much leisure to consume will effect his total wealth through firm profits.

- Lagrangian:

$$\mathcal{L} = h^{1-\beta}c^{\beta} + \lambda[w - P_c c - P_h h].$$

  FOCs imply (due to the Cobb-Douglas Utility):

$$c^{cons} = \frac{\beta w}{P_c},$$

$$h^{cons} = \frac{(1-\beta)w}{P_h}.$$

  This is Robinson's demand for coconuts and leisure.

- See G-3.2. We have nice indifference curves and an interior solution. The only note to make is the area below the curve to the right of T. Clearly, if Robinson chooses more than T hours of leisure, he will not work and profits will be zero. But since this is NOT interalized, there still seems to be a way to reach these bundles. In the market clearing phase, we will eliminate this possibility of landing here.

- The firm's problem is as we did above in our example. See G-3.3. The production frontier is a nice smooth curve and we can also plot isoprofit lines where tangency specifies the point of maximization.

- Finally market clearing says:

$$h^{cons} + h^{firm} = T,$$

$$c^{cons} = c^{firm} = \sqrt{h^{firm}}.$$

As usual, we only need one condition. Note that:

$$w = P_h T + \pi(P_c, P_h) = P_h T + \frac{P_c^2}{4P_h}.$$

So, let's try the labor condition:

$$\begin{aligned}
h^{cons} + h^{firm} &= T \\
\frac{(1-\beta)w}{P_h} + \left(\frac{P_c}{2P_h}\right)^2 &= T \\
(1-\beta)T + \frac{(1-\beta)P_c^2}{4P_h^2} + \frac{P_c^2}{4P_h^2} &= T \\
\frac{P_c^2}{P_h^2}\left(\frac{1-\beta}{4} + \frac{1}{4}\right) &= T - (1-\beta)T \\
\frac{P_c^2}{P_h^2}\frac{2-\beta}{4} &= \beta T \\
\frac{P_c^2}{P_h^2} &= \frac{4\beta T}{2-\beta} \\
\frac{P_c}{P_h} &= \sqrt{\frac{4\beta T}{2-\beta}}
\end{aligned}$$

We can then plug this ratio into the equations for $c^{cons*}, c^{firm*}, h^{cons*}, h^{firm*}$ and solve just in terms of the parameters of the model: β and T.

- See G-3.4 for a graph of the solution. Note how we transform the production set graph into an edgeworth box by moving to the left by T units and starting the consumer's origin there. Pretty slick.

3.2 Efficiency

- What do we mean by an efficient market outcome and is our Walrasian equilibrium efficient?

- **Definition:** Pareto Efficiency. A feasible allocation (x, y) is pareto optimal if no other feasible allocation, (x', y'), such that $x'_i \succeq_i x_i \; \forall \; i$ and $x'_i \succ x_i$ for at least one consumer i. Note we don't say anything about profit maximization. Just need to make the consumers as sweet as possible.

- In the edgeworth box analysis, it is clear that all bundles located in "De Lenz" will be pareto improving. Increasing the number of consumers in the problem makes things more difficult. It is clear from G-3.4 that the Walrasian equilibrium in the Crusoe problem is pareto efficient.

- All tangencies between indifference curves in an edgeworth box (pareto efficient allocations) make up the Pareto set. Given an endowment point, the subset of the Pareto set that is pareto improving, starting at the endowment point, is called the Contract Curve. See G-3.5.

3.3 First Welfare Theorem

- In general, the First Welfare Theorem (FWT) says that all Walrasian equilibria are pareto efficient. It is a formalization of Adam Smith's invisible hand and the efficiency of market outcomes.

- We impose a weak assumption regarding consumer's preferences: Local Non-Satiation. This means:

$$\text{for any } x_i \in \Re^L_+, \text{ and } \forall\, \epsilon > 0,\ \exists\, x'_i \in \Re^L_+ \ni x'_i \succ_i x_i \text{ and } ||x'_i - x_i|| < \epsilon.$$

  So strict monotonicity in just one good is enough to get LNS. Lexicographic preferences are also ok. We just can't have situations like "thick" indifference curves or satiation points.

- Statement of theorem: If consumer preferences are locally non-satiated and if (x^*, y^*, p) is a Walrasian equilibrium, then (x^*, y^*) is Pareto efficient.

4 Lecture 4: February 8, 2005

4.1 First Welfare Theorem

- If the consumer's preferences exhibit local non-satiation and if (x^*, y^*, p) is a Walrasian equilibrium, then (x^*, y^*) is Pareto efficient.

- Proof. Let (x^*, p) be a Walrasian equilibrium. Suppose there exists a feasible $\tilde{x}$ that Pareto dominates x^*. Thus,
$$\tilde{x}_i \succeq_i x_i^* \; \forall \; i,$$
and,
$$\tilde{x}_i \succ_i x_i^* \text{ for some } i.$$
If $\tilde{x}_i \succeq_i x_i^*$, then $p\tilde{x}_i \geq w_i$ by LNS. If $\tilde{x}_i \succ_i x_i^*$, then $p\tilde{x}_i > w_i$ by optimization. Thus,
$$\sum_i p\tilde{x}_i > \sum_i w_i = p \cdot \overline{\omega}.$$
But since $\tilde{x}$ is feasible, $\sum_i \tilde{x}_i \leq \overline{\omega}$, or $p \cdot \sum_i \tilde{x}_i \leq p \cdot \overline{\omega}$. This is a contradiction so the theorem must hold. Note that we did NOT need convexity of indifference curves, just LNS, a pretty weak assumption.

- Cobb-Douglas Example. Recall that with utility:
$$u_i = x_{1i}^{\alpha} x_{2i}^{1-\alpha},$$
$$\omega_1 = (1, 2), \;\; \omega_2 = (2, 1),$$
the price vector was:
$$\frac{p_1}{p_2} = \frac{\alpha}{1-\alpha},$$
and the allocations were:
$$x_1* = (2-\alpha, 2-\alpha), \;\; x_2^* = (1+\alpha, 1+\alpha).$$
See G-4.1. We see that the entire pareto set is along the 45 degree line. Thus our Walrasian equilibrium is Pareto efficient.

- How do you find the Pareto efficient set of allocations? Consider the problem:
$$Max \; u_1 = x_{11}^{\alpha} x_{21}^{1-\alpha},$$
subject to:
$$u_2 = x_{12}^{\alpha} x_{22}^{1-\alpha} \geq \overline{u},$$
$$x_{11} + x_{12} \leq 3,$$
$$x_{21} + x_{22} \leq 3.$$

Now by varying the value of $\overline{u}$ (between 0 and 3), we trace out the Pareto efficient set of allocation. We could solve this by setting up the lagrangian:

$$\mathcal{L} = x_{11}^{\alpha} x_{21}^{1-\alpha} + \lambda[(3 - x_{11})^{\alpha}(3 - x_{21})^{1-\alpha} - \overline{u}].$$

FOCs yield:

$$x_{11} = x_{21} = 3 - \overline{u},$$
$$x_{12} = 3 - x_{11} = \overline{u},$$
$$x_{22} = 3 - x_{21} = \overline{u}.$$

And this traces out the 45 degree line in G-4.1.

- Note that the PE set is also where the slope of the indifference curves are equal. This is another way of finding the Pareto efficient set. Consider:

$$x_{11}^{\alpha} x_{21}^{1-\alpha} = \overline{u}.$$

Totally differentiate:

$$\alpha x_{11}^{\alpha-1} x_{21}^{1-\alpha} dx_{11} + (1-\alpha) x_{11}^{\alpha} x_{21}^{-\alpha} dx_{21} = 0.$$

$$\frac{dx_{21}}{dx_{11}} = -\frac{\alpha}{1-\alpha}, \text{ along the 45 degree line.}$$

- So, to summarize:
  - (1) All Walrasian equilibrium are Pareto efficient.
  - (2) The result holds for both exchange economies and production economies.
  - (3) We only needed LNS to prove the theorem.
  - (4) Pareto efficiency is a WEAK concept, there are LOTS of them. No mention of distribution until second theorem.

4.2 The Second Welfare Theorem

- In general, the economy has many ways of distributing goods that are pareto efficient. What about distributional concerns?
- The second welfare theorem says that any Pareto efficient allocation can be attained by using an appropriate transfer of wealth with a decentralized market system. So if the economy wants to hit a certain Pareto efficient allocation (maybe right in the middle of an edgeworth box), we have the tools to put them on the right track.
- Redistribution policies are not necessarily in conflict with market efficiency. The equilibrium in a market depends crucially on the initial endowments.
- Think of an economy with L goods, I consumers, J firms and total endowments of the L goods:

$$\overline{\omega} = (\overline{\omega}_1, \ldots, \overline{\omega}_L).$$

Preferences are complete and transitive on $\Re^L_+$ and $Y_j \subset \Re^L$ are the firm's production sets.

- **Definition:** The allocation (x^*, y^*) and the price vector, p, form a "price equilibrium with transfers" if there exists wealth levels,

$$(w_1, w_2, \ldots, w_I),$$

such that,

$$\sum_i w_i = p \cdot \overline{\omega} + \sum_j p \cdot y_j^*$$

and:

  - (1) Firms maximize profits:

$$p \cdot y_j \leq p \cdot y_j^* \; \forall \; y_j \in Y_j \; \forall \; j = 1, \ldots, J.$$

  - (2) Consumers maximize utility:

$$x_i^* \succeq_i x_i \; \forall \; x_i \in B_i(p) = \{x_i \in \Re^L_+ | p \cdot x_i \leq w_i\}.$$

  - (3) Markets clear:

$$\sum_i x_{li}^* = \overline{\omega}_l + \sum_j y_{lj}^* \; \forall \; l = 1, \ldots, L.$$

So we have not said anything about initial endowments, we are simply relying on the fact that there are wealth levels that exist which satisify consumer and firm optimization as well as market clearing.

- Assumptions of the Second Welfare Theorem:

  - (1) Each consumer has convex, continuous and strictly monotonic preferences on $\Re^L_+$.
  - (2) Each firm's production set, Y_j, is convex.
  - (3) The economy has a strictly positive production vector: $\overline{\omega}_l + \sum_j y_j >> 0$.

The important assumption is convexity of preferences and the production set.

- **Theorem:** Second Welfare Theorem. If an economy satisfies the above assumptions, then if (x^*, y^*) is a Pareto efficient allocation, there exists a price vector $p >> 0$ such that (x^*, y^*, p) is a price equilibrium with transfers. So this is saying that we can allocate wealth in such a way that the market will do its thing and make us reach the Pareto efficient point (any Pareto efficient point we choose).

- See G-4.2 for a picture of how we could use transfers to get on the right path and let the market work to move us to our choice of x^*.

- Where does the price ratio come from? It comes from the tangency of the indifference curves. This is where convexity comes in. See G-4.3 for non-convex preferences. Here, we could move the economy to the right path but the two consumers would want different equilibrium allocations.

- The mathematics behind this is that the upper-contour sets of each individual's indifference curves must be convex so that there exists a separating hyperplane (the price vector) that we can squeeze in between them at the tangency point.

- See G-4.4 for the Robinson Crusoe economy. Note that the convexity assumption is strong. It rules out things like CRS production functions.

- Question: If a society views a particular Pareto efficient outcome as socially optimal, is it possible for a competitive market system to achieve that outcome? Yes, BUT only if any redistribution of wealth is possible. We would also need the entire society to agree on what is socially optimal (collective choice issues). Also, the social planner (say the gov't) must be able to redistribute wealth correctly and costlessly. Chances are less than good.

- So the goal of "Second Best Welfare Economics" might be to examine the trade-offs between efficiency and distributional goals.

Calculus Approach to Pareto Efficiency

- Assume consumers have utility functions $u_i(x_i)$ which are C^2, concave and strictly increasing in each good.

- Assume firms have technology given by a C^2, concave, tranformation function, F, such that the production set is defined as:

$$Y_j = \{y \in \Re^L : F_j(y) \leq 0\},$$

with:

$$\frac{\partial F_j}{\partial y_{lj}} > 0 \; \forall \; l = 1, \ldots, L.$$

- The transformation function might look like:

$$F(-h, c) = c - \sqrt{h} \leq 0.$$

Note that it is negative in the range where it is feasible. See G-4.5 The slope of the boundry is:

$$\frac{dc}{dh} = -\frac{\partial F/\partial h}{\partial F/\partial c},$$

by the implicit function theorem. This ratio is called the Marginal Rate of Tranformation (MRT).

- Pareto efficient allocations (x^*, y^*) are characterized as solutions to a constrained maximization problem for consumer 1:

$$Max_{x,y}\ u_1(x_1),$$

  such that:

  - (1) $u_i(x_i) \geq \overline{u}_i$, for $i = 2, \ldots, I$
  - (2) $\sum_i x_{li} \leq \overline{\omega}_l + \sum_j y_{lj}$ for $l = 1, \ldots, L$
  - (3) $F_j(y_j) \leq 0$ for $j = 1, \ldots, J$

  for some choices of $(\overline{u}_2, \overline{u}_3, \ldots, \overline{u}_I)$.

- Consider the lagrangian:

$$\mathcal{L} = u_1(x_1) + \sum_{i=2}^{I} \delta_i(u_i(x_i) - \overline{u}_i) - \sum_{l=1}^{L} \mu_l(\sum_{i=1}^{I} x_{li} - \overline{\omega}_l - \sum_{j=1}^{J} y_{lj}) - \sum_j \gamma_j F_j(y_j),$$

  where δ, μ, and γ are all the multipliers.

- Kuhn Tucker Conditions for Pareto Efficiency:

$$\frac{\partial \mathcal{L}}{\partial x_{l1}} : \ \frac{\partial u_1}{\partial x_{l1}} - \mu_l = 0, \text{ for } l = 1, \ldots, L.$$

$$\frac{\partial \mathcal{L}}{\partial x_{li}} : \ \delta_i \frac{\partial u_i}{\partial x_{li}} - \mu_l = 0, \text{ for } l = 1, \ldots, L, i = 2, \ldots, I.$$

$$\frac{\partial \mathcal{L}}{\partial y_{lj}} : \ \mu_l - \gamma_j \frac{\partial F_j}{\partial y_{lj}} = 0, \text{ for } l = 1, \ldots, L, j = 1, \ldots, J.$$

- From the KT conditions, we can show that:

  - (1)

$$\frac{\partial u_i / \partial x_{li}}{\partial u_i / \partial x_{l'i}} = \frac{\mu_l}{\mu_{l'}} = \frac{\partial u_{i'} / \partial x_{li'}}{\partial u_{i'} / \partial x_{l'i'}} \ \forall\, i, i', l, l'.$$

    Thus the MRS between any pair of goods is equalized, or goods are allocated optimally across consumers.

  - (2)

$$\frac{\partial u_i / \partial x_{li}}{\partial u_i / \partial x_{l'i}} = \frac{\mu_l}{\mu_{l'}} = \frac{\partial F_j / \partial y_{lj}}{\partial F_j / \partial y_{l'j}} \ \forall\, i, j, l, l'.$$

    Thus an optimal assortment of goods is produced.

  - (3)

$$\frac{\partial F_j / \partial y_{lj}}{\partial F_j / \partial y_{l'j}} = \frac{\mu_l}{\mu_{l'}} = \frac{\partial F_{j'} / \partial y_{lj'}}{\partial F_{j'} / \partial y_{l'j'}} \ \forall\, j, j', l, l'.$$

    Thus the MRT between any pair of goods is equalized, or production is efficient.

- So what's the link between all this calculus and the second welfare theorem? Well, the prices that need to be set to attain a Pareto efficient allocation (x^*, y^*) are the set of lagrange multipliers:

$$p = (\mu_1, \ldots, \mu_l).$$

  So the actual price of each of the goods equals its shadow price.

5 Lecture 5: February 10, 2005

5.1 Existence of an Equilibrium

- Is the concept of a decentralized market system guided by prices coherent? In the GE model, the interdependence of markets could cause problems for the existence of market clearing prices.

- Consider an exchange economy with I consumers and L goods. Preferences are continuous, STRICTLY convex, and strongly monotone. These are pretty strong assumptions.

- Endowments as usual:
$$\omega_i = (\omega_{1i}, \ldots, \omega_{Li}) \geq 0 \; \forall \; i.$$
Aggregate endowment is $\overline{\omega}_l$.

- In our consumer maximization problem, consumer i demands x_i^* such that:
$$x_i^* \succeq_i x_i \; \forall \; x_i \in B_i(p) = \{x_i \in \Re^L_+ \ni p \cdot x_i \leq p \cdot \omega_i\}.$$
See G-5.1. For non-convex preferences, it is possible for the offer curve to be going along and then jump to a new point when the price vector reaches a certain point. This means that the offer curve for the agent is not continuous and we may have a situation when the two offer curves do NOT cross. Continuity and convexity are key then for existence.

- For strictly convex preferences, the solution, $x_i^* = x_i(p, p \cdot \omega_i)$, is UNIQUE and the demand function is continuous and indeed a function and not just a correspondence.

- Define again a Walrasian equilibrium: An allocation $x^* = (x_1^*, \ldots, x_I^*)$ and a price vector, $p >> 0$, constitute a Walrasian equilibrium for an exchange economy if:
  - (1) Consumers maximize utility:
  $$x_i^* = x_i(p, p \cdot \omega_i) \; \forall \; i.$$
  - (2) All markets clear:
  $$\sum_i x_{li}^* = \sum_i \omega_{li}^* \; \forall \; l.$$

- **Definition:** The Excess Demand function of consumer i is:
$$Z_i(p) = (Z_{1i}(p), \ldots, Z_{Li}(p)),$$
where,
$$Z_{li}(p) = x_{li}(p, p \cdot \omega_i) - \omega_{li}.$$
So it's consumer i's net trade of good l.

- The aggregate excess demand function is thus:

$$Z(p) = (Z_1(p), \ldots, Z_L(p)),$$

where,

$$Z_i(p) = \sum_i x_{li}(p, p \cdot \omega_i) - \sum_i \omega_{li}.$$

- In our Cobb-Douglas example,

$$Z_1(p) = x_{11} + x_{12} - \omega_{11} - \omega_{12}$$

$$= \alpha(1 + 2(p_2/p_1)) + \alpha(2 + (p_1/p_2)) - 1 - 2 = \alpha(3 + 3(p_2/p_1)) - 3.$$

- If:

$$Z_l(p) > 0 \Longrightarrow \text{Demand } (D_l) > \text{Supply } (S_l).$$
$$Z_l(p) < 0 \Longrightarrow \text{Demand } (D_l) < \text{Supply } (S_l).$$
$$Z_l(p) = 0 \Longrightarrow \text{Demand } (D_l) = \text{Supply } (S_l).$$

- **Theorem:** So, a price vector, $p >> 0$, is a Walrasian equilibrium iff $Z(p) = 0$. So the existence of a solution to the GE problem is equivalent to the existence of a solution to the system of L equations:

$$Z_1(p) = 0, Z_2(p) = 0, \ldots, Z_L(p) = 0.$$

- Now consider 4 properties of Aggregate Excess Demand. Consider an exchange economy where consumer preferences are continuous, strictly convex, and strongly monotone, and where $\overline{\omega}_l > 0$ for each good l. Then aggregate excess demand function, $Z(p)$, is defined for all $p >> 0$ and satisfies:
  - (A1) $Z(p)$ is continuous.
  - (A2) $Z(\lambda p) = Z(p)\ \forall\ \lambda > 0$. $Hod(0)$.
  - (A3) $p \cdot Z(p) = 0\ \forall\ p$. Walras Law.
  - (A4) If $p^n \to \overline{p} \neq 0$ where $\overline{p}_l = 0$ for some good l, then the excess demand for at least one good must go to infinity.

- Notes on assumptions. (A1) follows from the strictly convex and continuous preferences. (A2) follows from the Walrasian demands being $hod(0)$ – absence of money illusion. (A3) says that everyone spends all their money. It further implies that if there are N markets and $N-1$ of them are in equilibrium then the N^{th} also must clear. This is the reason why we only need one market clearing condition to solve for the equilibrium price ratio.

- **Theorem:** Suppose $Z(p)$ is any function defined for all $p >> 0$ that satisfies the above four assumptions. Then the system of equations, $Z(p) = 0$, has a solution. Note that (A3) has the excess demands weighted by the prices. This must be zero, but that alone is not strong enough for existence since some markets (goods) may have excess supply and some have excess demand, but Walras law still holds. In addition to Walras you need the other 3 assumptions, and this guarantees a solution to our system of L equations.

- **Corollary:** An exchange economy with preferences that are continuous, strictly convex, and strictly monotone and with aggregate endowment $\overline{\omega} >> 0$ has a Walrasian equilibrium. Pretty strong assumptions.

- Proof for $L = 2$. We want to show that $\exists p = (p_1, p_2)$ with $Z(p) = 0$. Normalize the price of the second good to 1 so $p = (p_1, 1)$. By Walras law, we just need to show that one of these markets is in equilibrium to show they both are. So we know we have a Walrasian equilibrium if $Z_1(p_1, 1) = 0$. So proceed as follows.
Suppose we let $p_1 \to 0$. By (A4), the excess demand for "some" good must go to infinity. Can it be good 2? No, because $p_2 = 1$ and consumers have a finite amount of wealth. So it must be the excess demand for good 1, $Z_1(\cdot)$, that goes to infinity.
Now, let $p_1 \to \infty$. So $p = (p_1, 1) \to (\infty, 1)$. By (A2), we can renormalize the excess demand function so:
$$Z(p_1, 1) = \frac{1}{p_1} Z(p_1, 1) = Z(1, 1/p_1).$$
So by letting $p_1 \to \infty$, is equivalent to letting $p_2 \to 0$. Again, by the argument that some good's excess demand goes to infinity, we see that it must be good 2's excess demand that's blowing up. (Intuitively this also makes sense, but the argument is more subtle). Thus $Z_2(1, 1/p_1) \to \infty$. So by Walras law:
$$Z_1(1, 1/p_1) + Z_2(1, 1/p_1) = 0 \Longrightarrow Z_1(1, 1/p_1) = -\underbrace{Z_2(1, 1/p_1)}_{\to\infty}.$$
Thus $Z_1(\cdot)$ must be negative for large enough p_1. See G-5.2. Since the excess demand for good 1 crosses the x-axis for some price level, we have shown that the market for good 1 clears. By Walras, the market for good 2 clears. Thus we have a Walrasian equilibrium. Craziness.

- The proof for L goods is more tedious and is only outlined here. We start by defining a Unit Price Simplex:
$$\Delta^L = \{p \in \Re^L_+;\ \ni \sum_l p_l = 1\}.$$
So all prices are normalized to add to one. Thus the price set is compact. We define a function f such that:
$$f_l(p) = \frac{p_l + Z_l^+(p)}{\sum_{h=1}^L (p_l + Z_h^+(p))},$$
where $Z_l^+(p) = max\{Z_l(p), 0\}$.

- $f(\cdot)$ has no good intuition except that it scales up the prices in such a way that we can apply a fixed point theorem. Since the price space is compact, by showing that f has a fixed point (via Brouwers), ie $f(p^*) = p^*$, this price vector satisfies $Z(p^*) = 0$ and thus is a Walrasian equilibrium.

- Note that in all of this, we have been requiring that consumer preferences are continuous, strictly convex and strongly monotone ... strong assumptions. An economy may have a Walrasian equilibrium when these conditions fail, but we cannot guarantee existence.

6 Lecture 6: February 15, 2005

6.1 Uniqueness

- What does a multiplicity of equilibria look like? See G-6.1. x_1 and x_2 can both be supported as Pareto equilibria. However person 1 likes x_2 better. The GE model will not make an exact prediction when there are multiple equilibria. If we had uniqueness, we would have a much easier time figuring out what was happening. It's also hard (or impossible) to do comparative statics when there is more than one equilibrium.

- Consider an exchange economy with I consumers and L goods. Assume preferences are continuous, strictly convex and strongly monotone. Recall the aggregate excess demand function is $Z(p) = (Z_1(p), \ldots, Z_L(p))$ with:

$$Z_l(p) = \sum_i x_{li}(p, w_i) - \sum_i \omega_{li}.$$

  A price vector, p, is a Walrasian equilibrium iff $Z(p) = 0$.

- Recall our 4 assumptions from last time: (1) $Z(p)$ continuous, (2) $Z(\lambda p) = Z(p)$, (3) $p \cdot Z(p) = 0$ for all p, (4) if $p^n \to \overline{p} \neq 0$ where $\overline{p}_l = 0$ for some good l then the excess demand for at least one good must go to infinity.

- Consider a 2 good example. We have shown that a price vector, $p = (p_1, 1)$ is a Walrasian equilibrium iff $Z_1(p_1, 1) = 0$. See G-6.2. In panel (a), we have just one unique equilibrium. Panel (b) has three.

- Where do the three equilibra come from? Consider differentiating $Z_1(p_1, 1)$:

$$\frac{\partial Z_1(p_1, 1)}{\partial p_1} = \sum_{i=1}^{I} \frac{\partial x_{1i}(p, w_i)}{\partial p_1} + \underbrace{\frac{\partial x_{1i}(p, w_i)}{\partial w_i} \cdot \omega_{1i}}_{Wealth\ Effects}.$$

  Because $w_i = p \cdot \omega_i$. Note that these wealth effects could be positive, negative, or zero so it is definitely possible for the excess demand function to slope up.

- Now consider G-6.2 panel (c) and (d). In (c) we have a countably infinite number of equilibria and in (d) we have a continuum (uncountable infinite number) of equilibria.

- In G-6.3 we do a policy experiment which shows that if there is an external shock to the economy which shifts the excess demand function, equilibria may bifurcate or even disappear completely! We could look at just local ranges around an equilibrium and do comparative statics from a given starting point, but what if we don't know where we are starting from?

- **Definition:** In panel (c), the Walrasian equilibrium are locally isolated, or locally unique. A Walrasian equilibrium price vector, p^*, is locally unique if there exists an $\epsilon > 0$ such that if $p \neq p^*$ and $||p - p^*|| < \epsilon$, then $Z(p) \neq 0$.

- If we select a particular locally unique equilibrium, then we can consider how the equilibrium changes locally when some parameter of the economy changes. This is called local comparative statics.

- When are Walrasian equilibria locally unique? Consider a normalized price vector,

$$p = (p_1, p_2, \ldots, p_{L-1}, 1).$$

So we have normalized the L^{th} good. Thus p is a Walrasian equilibrium iff $\hat{Z}(p) = 0$ where,

$$\hat{Z}(p) = (Z_1(p), \ldots, Z_{L-1}(p)).$$

We only need the first $L-1$ goods since by Walras Law, if they are all clearing, then the market for the L^{th} good also clears.

- Consider the following matrix of price effects:

$$D\hat{Z}(p) = \begin{bmatrix} \dfrac{\partial Z_1(p)}{\partial p_1} & \cdots & \dfrac{\partial Z_1(p)}{\partial p_{L-1}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial Z_{L-1}(p)}{\partial p_1} & \cdots & \dfrac{\partial Z_{L-1}(p)}{\partial p_{L-1}} \end{bmatrix}.$$

- **Definition:** So in the case of 2 goods, $D\hat{Z}(p)$ is just the derivative of Z_1 with respect to p_1. Claim that an equilibrium price vector $p = (p_1, \ldots, p_{L-1}, 1)$ is REGULAR if the matrix $D\hat{Z}(p)$ is invertible. If every equilibrium price vector of an economy is regular, we say the economy itself is regular.

- Note that $D\hat{Z}(p)$ is an $L - 1xL - 1$ matrix. If we kept in the last good, we would have a singular matrix (again by Walras Law). Hence:

$$D\hat{Z}(p) \text{ invertible } \Leftrightarrow rank(D\hat{Z}(p)) = L - 1 \Leftrightarrow |D\hat{Z}(p)| \neq 0.$$

- In panel (b), $D\hat{Z}(p) \neq 0$ at all points that the excess demand function crossed the axis so that economy is regular.

- In panel (d), $D\hat{Z}(p) = 0$ at all price vectors so the economy is not regular.

- **Proposition** Two points:

  - (1) Any regular equilibrium price vector $p = (p_1, \ldots, p_{L-1}, 1)$ is locally unique.
  - (2) A regular economy has a finite number of equilibrium price vectors.

  Proof of point 1: Let p^* be a regular equilibrium. Then $\hat{Z}(p^*) = 0$. Consider a nearby price vector $p^* + \Delta p$ with $\Delta p \neq 0$. Since $D\hat{Z}(p)$ is invertible, $D\hat{Z}(p^*) \cdot \Delta p \neq 0$. Thus,

$$\hat{Z}(p^* + \Delta p) \approx \hat{Z}(p*) + D\hat{Z}(p^*)\Delta p = 0 + D\hat{Z}(p^*)\Delta p \neq 0.$$

  Thus $p^* + \Delta p$ is NOT an equilibrium.

- In general, we might think that there will be an odd number of regular equilibria. Existance tells us there will be one. Every time $Z(p)$ comes up across the x-axis, it must return since it must eventually turn negative.

- **Theorem:** The Index Theorem. Any regular economy has an odd number of Walrasian equilibria (and definitely has at least one).

- While knowing how many Walrasian equilibrium there are is not that useful, knowing existance and uniqueness is helpful and if the equilibrium is not unique, is it locally unique?

- So how likely is it that an economy has a continuum of WE as in G-6.2(d). We will show that it is not likely at all.

- **Definition:** A set $A \in \Re^n$ is generic if the complement of the set, $A^c = \Re^n - A$, has Lebesgue measure of 0. In other words, if the probability of observing something outside of the set if zero. Think of a point or a line in a continuous distribution. The set besides the point or line is generic.

- So now we fix preferences, $\succeq_i$ for $i = 1, \ldots, I$, and consider generic sets of endowment vectors:
$$\omega = (\omega_1, \ldots, \omega_L) \in \Re^{LxI}_{++}.$$
So bad things can happen (like G-6.2(d)), but they have zero probability.

- **Proposition** The set of all endowments $\omega \in \Re^{LxI}_{++}$ such that
$$\{\succeq_i, \omega_i, i = 1, \ldots, I\}$$
is a regular economy, is generic. That is, the set of endowments, ω, such that the associated economy has an infinite number of WE has measure 0. So G-6.2(d) is NOT typical.

- Conclusion: Almost all economies have a finite number of Walrasian equilibria. These equilibra are locally isolated so we can do local comparative statics.

- The example we did where $u_1 = x_{11} + \dfrac{1}{x_{21}}$ and $u_2 = x_{22} - \dfrac{1}{x_{12}}$ with endowments $\omega_1 = (4, 0)$ and $\omega_2 = (0, 4)$ was rigged!! For general endowments, we would not get the result of a continuum of equilbria. The endowments choosen gave us the strange outcome – not typical.

7 Lecture 7: February 17, 2005

7.1 Global Uniqueness

- What conditions do we need in the GE model to get global uniqueness? It turns out they are pretty strong assumptions. There are two approaches:
  - (1) Gross Substitutes Property.
  - (2) Weak Axiom.

  Both properties are ways to make sure that the wealth effects are not large enough to overwhelm the substitution effects so that the aggregate excess demand curve is always downward sloping (at least at the equilibria).

- In general, we get multiplicity when wealth effects are too large. Also, when we have complementarities in the markets, multiplicity often arises. Consider:

$$u_1 = u_2 = min\{x_1, x_2\},$$

  so both consumers have perfect complement type utility functions. See G-7.1. It is clear that drawing separating price vectors is fairly easy in this case.

Gross Substitution

- So consider an exchange economy with I consumers and L goods. Preferences are continuous, strongly monotone and strictly convex. The total endowment of all goods is strictly positive.
- Aggregate excess demand, $Z(p)$ satisfies assumptions A1-A4. We also assume $Z(p)$ is C^1 for $p >> 0$.
- So what do we need for global uniqueness? One way is for all the goods to be gross substitutes.
- **Definition:** 2 goods, l and $\tilde{l}$ are gross substitutes (GS) at a price vector, p, if:

$$\frac{\partial Z_{\tilde{l}}}{\partial p_l} > 0, \text{ and } \frac{\partial Z_l}{\partial p_{\tilde{l}}} > 0.$$

  So aggregate demand for good l is increasing in the price of good $\tilde{l}$ and vice versa.

- **Definition:** An economy has the GS property if $\forall\ l \neq \tilde{l}$ and $p >> 0$,

$$\frac{\partial Z_{\tilde{l}}}{\partial p_l} > 0.$$

  Note this is an uncompensated demand property. We are not saying anything about what happens to wealth (ie, we don't compensate consumers for the change in prices).

- So in the matrix of price effects, $D\hat{Z}(p)$, an $L-1xL-1$ matrix, all the off diagonal elements are strictly positive. All the main diagonal entries are negative by the following proof.
  Proof: By (A2),
  $$Z_l(\lambda p_1, \ldots, \lambda p_L) = Z_l(p_1, \ldots, p_L).$$
  Differentiate this expression with respect to λ:
  $$\sum_{\tilde{l}=1}^{L} \frac{\partial Z_l(p)}{\partial p_{\tilde{l}}} p_{\tilde{l}} = 0.$$
  Pick out just the l^{th} good:
  $$\frac{\partial Z_l(p)}{\partial p_l} p_l + \sum_{\tilde{l} \neq l} \frac{\partial Z_l(p)}{\partial p_{\tilde{l}}} p_{\tilde{l}} = 0.$$
  Or,
  $$\frac{\partial Z_l(p)}{\partial p_l} p_l = \underbrace{-\frac{1}{p_l} \sum_{\tilde{l} \neq l} \frac{\partial Z_l(p)}{\partial p_{\tilde{l}}} p_{\tilde{l}}}_{negative}.$$

- **Proposition** Global Uniqueness (Sufficient condition). If the aggregate excess demand function, $Z(p)$, for an exchange economy satisfies A1-A4 and the GS property, then the economy has one normalized equilibrium p^*.
  Proof: Suppose p and p' are 2 Walrasian equilibrium such that $p \neq \lambda p'$. Thus one price vector is not a multiple of the other. We can always find the normalization of p and p' such that $p'_l = p_l$ for one good l, and $p'_{\tilde{l}} \geq p_{\tilde{l}}$ $\forall$ other goods. Now, take a "walk" of $L-1$ steps from p up to p'. Think of increasing the prices for at least one and keeping the others the same from p to p'. At each step:
  $$\frac{\partial Z_l(p)}{\partial p_{\tilde{l}}} \geq 0,$$
  for each good. So we're thinking about the excess demand for good l and looking at the effects of the new prices. Since the price of good l has stayed the same, the diagonal entry is the same. Since $p \neq \lambda p'$, there must exist at least some entry on the good l's row of $D\hat{Z}(p)$ that is strictly positive. ((The elements where $p_l = p'_l$ will be the same for both matricies)). So if $Z(p) = 0$ then $Z(p') > 0$ which contradictes p' being an equilibrium price vector.

- A note on the "normalization". Just scale up the price vector in such a way so that all the prices are at least as big as the other set of prices and one is exactly equal to the other price.

- When does the GS property hold? Cobb Douglas utility functions of the form:

$$u_i(x_i) = x_{1i}^{\alpha_{1i}} x_{2i}^{\alpha_{2i}} \cdots x_{Li}^{\alpha_{Li}},$$

with $\sum_l \alpha_{li} = 1$ satisfies the GS property. For each consumer i, the Walrasian demand is:

$$x_{li}(p, w_i) = \frac{\alpha_{li} w_i}{p_l}.$$

Excess demand for good l is:

$$Z_l(p) = \sum_{i=1}^{I} \frac{\alpha_{li} w_i}{p_l} - \sum_{i=1}^{I} \omega_{il} = \sum_{\tilde{l}=1}^{L} \sum_{i=1}^{I} \frac{\alpha_{li}(p_{\tilde{l}} \omega_{\tilde{l}i})}{p_l} - \sum_{i=1}^{I} \omega_{il}.$$

Thus,

$$\frac{\partial Z_l(p)}{\partial p_{\tilde{l}}} = \sum_i \frac{\alpha_{li} \omega_{\tilde{l}i}}{p_l} > 0 \text{ if } \omega_{\tilde{l}i} > 0 \text{ for some } i.$$

So this economy has a unique equilibrium if for each good, there is a strictly positive endowment of that good by some consumer.

- We should keep in mind that the basic condition for the GS property is for the $D\hat{Z}(p)$ matrix to be negative definite with negative diagonal entries and positive off diagonal entries.

- Note that the CES utility function also satisfies the GS property and no restrictions on endowments are needed.

- Finally, note that the GS property is additive over individual excess demand functions. So if it is satisfies for all i, it is satisfied for the entire economy.

Weak Axiom

- An aggregate demand function, $Z(p)$, satisfies the weak axiom (WA) if for any pair of price vectors, p and p':

$$Z(p) \neq Z(p') \text{ and } p \cdot Z(p') \leq 0 \Longrightarrow p' \cdot Z(p) > 0.$$

So this says we have demand different at the two price vectors and the bundle $Z(p')$ is affordable at prices p, but the consumer still chooses $Z(p)$ (since $Z(p) = 0$). This means that $Z(p) \succeq Z(p')$. Given this, it must be the case that $Z(p)$ is not affordable at p' because otherwise the consumer would have bought it! But he doesn't. He buys $Z(p')$. The point is that if $Z(p) \succeq Z(p')$ at prices p, he must have the same preferences at p'. The consumer must be consistent.

- Law of Demand and the Weak Axiom. Suppose p and p' are price vectors with $Z(p) \neq Z(p')$ and $p \cdot Z(p') = 0$. Then the WA implies:

$$(p' - p) \cdot [Z(p') - Z(p)] < 0,$$

or the demand curve must be downward sloping!

- The WA always holds for an individual excess demand function but it is NOT an additive property like the GS property is. To impose it on the aggregate $Z(p)$ is strong.
- **Proposition** If the aggregate excess demand function satisfies A1-A4 and the WA, and if the economy is regular, then the economy has one normalized equilibrium p^*.
- When does the WA hold? Homothetic preferences + constant proportion endowments will make $Z(p)$ satisfy the WA. CD and CES utility functions are both homothetic.
- The freeness of the GE model (any preferences, any endowments) means that strong conditions must be imposed to guarantee uniqueness. Note that WA does not imply GS and vice versa. Both the conditions imply that substitution effects are large enough (larger than the wealth effects) to insure a unique equilibrium.
- Note that the WA does NOT rule out the possibility of a continuum of equilibria – you need generic endowments for that.

Summary So Far...

Assumptions so far in the course:

- (1) Weak: local nonsatiation gave us the First Welfare Theorem – Walrasian equilibrium are pareto efficient.
- (2) Moderate: continuity, strong monotonicity and strict convexity gave us the Second Welfare Theorem, existance, and equilibria that are locally unique and finite in number for generic endowments.
- (3) Crazy Strong: gross substitution property and the weak axiom gives us global uniqueness.

Notes on CD and CES Utility Functions

7.2 Cobb-Douglas Utility

- Consider a utility function of the form:

$$u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}.$$

- Assuming we have an exchange economy, wealth is:

$$w = \omega_1 p_1 + \omega_2 p_2.$$

- Lagrangian:

$$\mathcal{L} = x_1^{\alpha} x_2^{\beta} + \lambda(w - x_1 p_1 - x_2 p_2).$$

- First Order Conditions:

$$\begin{aligned}
(x_1): &\quad \alpha x_1^{\alpha-1} x_2^{\beta} = \lambda p_1. \\
(x_2): &\quad \beta x_1^{\alpha} x_2^{\beta-1} = \lambda p_2. \\
(\lambda): &\quad w = x_1 p_1 - x_2 p_2.
\end{aligned}$$

- Divide the first condition by the second:

$$\frac{\alpha}{\beta}\left(\frac{x_2}{x_1}\right) = \frac{p_1}{p_2}.$$

- Solve for x_2:

$$x_2 = \frac{\beta p_1 x_1}{\alpha p_2}.$$

- Substitute in budget constraint: $x_1 = (w - x_2 p_2)/p_1$:

$$x_2 = \frac{\beta p_1 (w - x_2 p_2)}{\alpha p_2 p_1}.$$

- Solve for x_2:

$$\begin{aligned}
x_2 &= \frac{\beta p_1(w - x_2p_2)}{\alpha p_2 p_1} \\
&= \frac{\beta w}{\alpha p_2} - \frac{\beta x_2 p_2)}{\alpha p_2} \\
&= \frac{\beta w}{\alpha p_2} - \frac{\beta x_2)}{\alpha} \\
x_2(1 + \frac{\beta}{\alpha}) &= \frac{\beta w}{\alpha p_2} \\
x_2 \frac{\alpha + \beta}{\alpha} &= \frac{\beta w}{\alpha p_2} \\
x_2 &= \frac{\alpha \beta w}{(\alpha + \beta)\alpha p_2} \\
x_2 &= \frac{\beta w}{(\alpha + \beta) p_2}
\end{aligned}$$

- And a similar method results for x_1. So:

$$x_1 = \frac{\alpha w}{(\alpha + \beta)p_1}.$$

$$x_2 = \frac{\beta w}{(\alpha + \beta)p_2}.$$

7.3 Constant Elasticity of Substitution Utility

- Consider a utility function of the form:

$$u(x_1, x_2) = (ax_1^\rho + bx_2^\rho)^{1/\rho}.$$

- Assuming we have an exchange economy, wealth is:

$$w = \omega_1 p_1 + \omega_2 p_2.$$

- Lagrangian:

$$\mathcal{L} = (ax_1^\rho + bx_2^\rho)^{1/\rho} + \lambda(w - x_1p_1 - x_2p_2).$$

- First Order Conditions:

$$(x_1) : \frac{1}{\rho}(ax_1^\rho + bx_2^\rho)^{\rho - 1} \rho a x_1^{\rho - 1} = \lambda p_1$$

$$(x_2) : \frac{1}{\rho}(ax_1^\rho + bx_2^\rho)^{\rho - 1} \rho b x_2^{\rho - 1} = \lambda p_2$$

$$(\lambda): \quad w = x_1 p_1 - x_2 p_2.$$

- Divide the first condition by the second:

$$\frac{a}{b}\left(\frac{x_1}{x_2}\right)^{\rho-1} = \frac{p_1}{p_2}.$$

- Solve for x_1:

$$x_1 = \left(\frac{bp_1}{ap_2}\right)^{1/(\rho-1)} x_2.$$

- Substitute in budget constraint: $x_2 = (w - x_1 p_1)/p_2$ and let $\delta = 1/(\rho - 1)$:

$$x_1 = \left(\frac{bp_1}{ap_2}\right)^{\delta} \frac{w - x_1 p_1}{p_2}.$$

- Solve for x_1:

$$\begin{aligned}
x_1 &= \left(\frac{bp_1}{ap_2}\right)^{\delta} \frac{w - x_1 p_1}{p_2} \\
x_1\left(1 + \frac{p_1}{p_2}\left(\frac{bp_1}{ap_2}\right)^{\delta}\right) &= \left(\frac{bp_1}{ap_2}\right)^{\delta} \frac{w}{p_2} \\
x_1 &= \frac{\left(\frac{bp_1}{ap_2}\right)^{\delta} \frac{w}{p_2}}{1 + \frac{p_1}{p_2}\left(\frac{bp_1}{ap_2}\right)^{\delta}} \\
&= \frac{\left(\frac{bp_1}{ap_2}\right)^{\delta} \frac{w}{p_2}}{1 + \frac{b^{\delta} p_1^{1+\delta}}{a^{\delta} p_2^{1+\delta}}} \\
&= \frac{\left(\frac{bp_1}{ap_2}\right)^{\delta} \frac{w}{p_2}}{\frac{a^{\delta} p_2^{1+\delta} + b^{\delta} p_1^{1+\delta}}{a^{\delta} p_2^{1+\delta}}} \\
&= \frac{\left(\frac{bp_1}{ap_2}\right)^{\delta} \frac{w}{p_2} [a^{\delta} p_2^{1+\delta}]}{a^{\delta} p_2^{1+\delta} + b^{\delta} p_1^{1+\delta}} \\
&= \frac{w b^{\delta} p_1^{\delta}}{a^{\delta} p_2^{1+\delta} + b^{\delta} p_1^{1+\delta}} \\
&= \frac{w p_1^{\delta}}{p_1^{1+\delta} + (a/b)^{\delta} p_2^{1+\delta}}
\end{aligned}$$

- And a similar method results for x_2. So:

$$x_1 = \frac{w p_1^{\delta}}{p_1^{1+\delta} + (a/b)^{\delta} p_2^{1+\delta}}.$$

$$x_2 = \frac{w p_2^{\delta}}{(b/a)^{\delta} p_1^{1+\delta} + p_2^{1+\delta}}.$$

8 Lecture 8: February 22, 2005

8.1 Stability Analysis

- In this section, we are trying to find out how prices "get to" their equilibrium. Are supply and demand forces the whole story? If we are at p^* and then something moves us away from equilibrium, will the market forces push us back to equilibrium? What happens if there are multiple equilibria?

- See G-8.1. If $p < p^*$, then $Z(p) > 0$ and if $p > p^*$, then $Z(p) < 0$.

- There are several criticisms of this type of analysis. Many ask, "who is in charge of adjusting these prices?" Who's behavior are we modelling? We might think that there is a game theoretic model behind all of this. Walras created a ficticous "Walrasian Auctioneer" who runs a centralized trading process to determine a market clearing price.

- **Proposition** Suppose $p(t) = (p_1(t), \ldots, p_{L-1}(t), 1)$ is the price adjustment path as a function of time. We normalize the L^{th} good. Then suppose we say that:

$$\frac{dp_l(t)}{dt} = \dot{p}_l = Z_l(p), \text{ for } l = 1, \ldots, L-1.$$

With the initial condition that $p(0) = (p_1^0, p_2^0, \ldots, p_{L-1}^0, 1)$. So what are we saying? We are hypothesizing that prices change in proportion to the excess demand of that good. This makes intuitive sense because of the dynamics in G-8.1. If $Z_l(p) > 0$, then there should be pressure on p_l to rise or $\dot{p}_l > 0$. At equilibrium,

$$\frac{dp_l(t)}{dt} = \dot{p}_l = Z_l(p) = 0 \; \forall \; l.$$

So we have a system of $L-1$ differential equations. $p(t)$ solves this system. Note that mathematically, there is nothing behind the time derivative of the price path equal to the excess demand. It just makes good economic sense.

- In the two good case,

$$\frac{dp_1(t)}{dt} = Z_1(p_1(t), 1),$$
$$p(0) = (p_1^0, 1).$$

Then $p_1(t)$ is a function that solves this first order differential equation. The price path is $p(t) = (p_1(t), 1)$.

- **Fact:** If $Z(p)$ is C^1 in p for $p >> 0$, then for any $p_0 >> 0$, $\exists$ a (locally defined) unique continuous function $p(t)$ that satisfies the system of differential equations and has $p(0) = p^0$.

- So stability analysis considers the behavior of $p(t)$ over time starting from any initial price, $p(0) = p^0$. We consider what happens to $p(t)$ as t goes to infinity. Does $p(t)$

converge to some price vector? Even if we don't know the functional form of $p(t)$, we are really only interested in the movement towards or away from equilibria. Does p approach p^*?

- See G-8.2. So for any $p(0)$, the price path converges to p^*. Note the downward sloping excess demand function. To the left of p^*, $Z(p) > 0$, so $\dot{p} > 0$.

- **Definition:** A price, p, is an equilibrium state of the system of differential equations if:
$$Z_l(p) = 0 \Longrightarrow \dot{p}_l = 0 \; \forall \; l.$$

- **Definition:** The price $p(t)$ converges to p^* as t approaches infinity if:
$$\forall \; \epsilon > 0, \; \exists \; \hat{t} \ni ||p(t) - p^*|| < \epsilon \text{ whenever } t > \hat{t}.$$

- **Definition:** An equilibrium state p^* is globally stable if $p(t)$ converges to p^* as t approaches infinity regardless of p^0. See G-8.2.

- **Definition:** An equilibrium state p^* is locally stable if,
$$\exists \; \epsilon > 0, \ni \; \forall \; p^0 \text{ with } ||p^0 - p^*|| < \epsilon, \Rightarrow p(t) \text{ converges to } p^* \text{ as } t \to \infty.$$
So there is some neighborhood around p^* that is convergent.

- See G-8.3 for two locally stable and one unstable equilibrium. An economy is called a stable system if all initial price vectors converge to SOME equililbrium (not necessarily unique). See Scarf's problem for a counterexample.

- This whole analysis gives us a way to evaluate equilibria. We might say that stable equilibria are plausible and globally stable equilibria are inevitable. Any unstable equilibrium is not plausible because any jolt to the economy will send us away from that point.

- **Proposition** Consider an economy with aggregate excess demand function $Z(p)$ satisfying A1 - A4. Suppose the economy satisfies EITHER the gross substitute property (GS) or the weak axiom (WA). Then the unique Walrasian equilibrium, p^*, is globally stable. So:
$$\text{Existence (A1-A4)} \quad + \quad \text{Uniqueness (GS or WA)} \quad = \quad \text{Globally Stability.}$$

- We can trace out the proof of this assumption for the $L = 3$ case using a phase diagram. See G-8.4. Suppose $p = (p_1, p_2, 1)$ so,
$$p(t) = (p_1(t), p_2(t), 1).$$

Then if $Z_1(p_1, p_2, 1) = 0$ at equilibrium,

$$\frac{dp_2}{dp_1} = -\frac{\overbrace{\partial Z_1/\partial p_1}^{-}}{\underbrace{\partial Z_1/\partial p_2}_{+}} > 0.$$

Also if $Z_2(p_1, p_2, 1) = 0$ at equilibrium,

$$\frac{dp_2}{dp_1} = -\frac{\overbrace{\partial Z_2/\partial p_1}^{+}}{\underbrace{\partial Z_2/\partial p_2}_{-}} > 0.$$

Both of these follow from the Implicit Function Theorem. So, the $\dot{p}_l = 0$ lines, which correspond to the $Z_l(p) = 0$, are upward sloping. See the Takayama paper for the reason why they cross once as shown. It is clear that above the $\dot{p}_1 = 0$ line, p_2 is higher, so this causes excess demand for good 1 which pushes the price of good 1 up or $\dot{p}_1 > 0$. The same analysis shows that the system is globally stable with a unique equilibrium at (p_1^*, p_2^*).

- See problem set 4 for a counterexample by Scarf. He shows that while there is a unique equilibrium price, the economy is not a stable system. The price paths never converge. The reason is that there are NO substitution effects (only income effects). Supply and demand forces are not strong enough.
- Conclusion:
  - (1) Stability is a robustness test: unstable equilibria are implausible.
  - (2) The GS property and the WA imply that the Walrasian price dynamics converge to a unique equilibrium.
  - (3) To get convergence we need sufficiently strong substitution effects.

9 Lecture 9: March 1, 2005

9.1 Foundations of a Competitive Equilibrium

- So far we have included a price mechanism for a means of exchange. How restrictive is this assumption? Are plausible allocations missed by the competitive model?

- **Definition:** A barter equilibrium or a voluntary exchange equilibrium is any feasible allocation which results from mutually advantageous trades between agents such that at equilibrium, no more advantageous trades are possible.

- This type of system comes out of cooperative game theory where coalitions of agents can take actions together. For example, a group of agents could break away from a price-based market system and just trade among their own endowments. The theory just considers what payoffs are plausible given the collection of coalitions.

- Setup of the model. L goods, I consumers. Preferences are continuous, strictly convex, strictly monotone. Total endowment, $\overline{\omega} >> 0$.

- **Definition:** Any subset $S \subseteq I$ of consumers is called a coalition. $S = I$ is the Grand Coalition.

- Key Point. Any coalition of agents can agree to trade just among themselves. What allocations might arise from this system?

- **Definition:** A feasible allocation x^* is blocked by a coalition S if we can find a consumption bundle x_i for each i in S such that:

  - (1) $x_i \succeq_i x_i^* \; \forall \, i \in S$.
  - (2) $x_i \succ_i x_i^*$ for some $i \in S$.
  - (3) $\sum_{i \in S} x_i = \sum_{i \in S} \omega_i$.

  So this last condition is what makes this different from Pareto efficiency. The blocking bundle must be feasible for the coalition.

- An allocation is blocked if at least one coalition blocks it.

- **Definition:** The core of an economy is the set of all feasible allocations that are not blocked. This will be the set of plausible allocations in our new barter economy.

- **Proposition** Core allocations are Pareto efficient.
  Proof: If a feasible allocation, x^*, is in the core, then no coalition blocks it, including the grand coalition. This is the exact definition of Pareto efficiency. Thus x^* is Pareto efficient.

- **Proposition** The core contains every Walrasian equilibrium allocation of an economy.
  Proof: Let x^* be a Walrasian equilibrium that is NOT in the core. Thus there is some coalition that blocks it. Thus the blocking bundle, $\tilde{x}$ satisfies:

- (1) $\tilde{x}_i \succeq_i x_i^* \; \forall \, i \in S \Rightarrow p \cdot \tilde{x}_i \geq p \cdot x_i^*$.
- (2) $x_i \succ_i x_i^*$ for some $i \in S \Rightarrow p \cdot \tilde{x}_i > p \cdot x_i^*$.

(1) and (2) together gives:

$$\sum_{i \in S} p \cdot \tilde{x}_i > \sum_{i \in S} p \cdot x_i^* = \sum_{i \in S} p \cdot \omega_i.$$

But this implies $\sum_{i \in S} p \cdot \tilde{x}_i > \sum_{i \in S} p \cdot \omega_i$, or:

$$\sum_{i \in S} \tilde{x}_i > \sum_{i \in S} \omega_i$$

which means that the blocking bundle is NOT feasible. Thus we have a contradiction which proves the proposition.

- Note that for $I = 2$, there are three coalitions. $\{1\}$ can block anything below his own indifference curve at the endowment, and the same for $\{2\}$. The coalition of $\{1, 2\} = I$ blocks anything off the pareto efficient set. See G-9.1.

- So we have shown that
$$WE \subseteq Core \subseteq PE.$$
Thus price taking equilibria are consistent with the idea of voluntary exchange, but in the core, we might have equilibria that are not Walrasian equilibrium.

- **Proposition** Edgeworth (1881). As an economy grows to include more and more consumers, the only outcomes that can remain in the core are Walrasian equilibria! This is because that with more consumers, there are more possible coalitions which cause the set of feasible allocations to become smaller and smaller. For a very large economy, the set of equilibria in a voluntary exchange economy are the SAME as those in a competitive economy.

- **Definition:** Suppose there are H types of consumers and each type is described by a preference relation and a fixed endowment. Assume the usual assumptions on the preferences and each type has a strictly positive endowment vector.

- **Definition:** Assume there are exactly N consumers of each type. This is called an N-replica economy. Thus there are NxH consumers with aggregate endowment:
$$\overline{\omega} = N \sum_h \omega_h.$$
As N increases, the economy gets larger but we still have the same relative proportion of types.

- An allocation for an N-replica economy is:
$$x = ((x_{11}, x_{12}, \ldots, x_{1N}); (x_{21}, x_{22}, \ldots, x_{2N}); \ldots; (x_{H1}, x_{H2}, \ldots, x_{HN})).$$

- **Proposition** Equal treatment. If an allocation x is in the core of an N-replica economy, then consumers of the same type must have the same bundle:

$$x_{h1} = x_{h2} = \cdots = x_{hN} \ \forall \ h.$$

Thus we can describe the core allocation by the allocation of each type: $x^* = (x_1, \ldots, x_H)$. Proof: Let x^* be in the core of an N-replica economy so the bundles for the type h consumers are $\{x^*_{h1}, \ldots, x^*_{hN}\}$. Suppose that these bundles are NOT all equal for each type h. Let x^*_{h1} be the worst off type h consumer. So $x^*_{hn} \succeq x^*_{h1} \ \forall \ n$ and $\forall \ h$. Let a coalition S be the coalition of the worst off consumer of each type. Give a bundle $\hat{x}$ to S such that:

$$\hat{x}_h = \frac{1}{N} \sum_{n=1}^{N} x^*_{hn}.$$

Clearly,

$$\hat{x}_{h1} \succeq_h x^*_{h1},$$

because getting the average allocation among your type must be at least as good as getting the lowest amount. Now is $\hat{x}$ feasible? Consider:

$$\begin{aligned}
\sum_{h=1}^{H} \hat{x}_h &= \frac{1}{N}[\sum_{h=1}^{H} \sum_{n=1}^{N} x^*_{hn}] \\
&= \frac{1}{N}\overline{\omega} \\
&= \frac{1}{N} N \sum_{h=1}^{H} \omega_h \\
&= \sum_{h=1}^{H} \omega_h
\end{aligned}$$

So since $\hat{x}$ is prefered by the coalition S, and $\hat{x}$ is feasible, then x^* cannot be in the core if the bundles are different for consumers of the same type. Thus there must be equal treatment.

- **Lemma** Shrinking Core. The core of the N-replica economy contains the core of the $N+1$-replica economy. If C_N is the core of an N-replica:

$$C_1 \supseteq C_2 \supseteq \cdots \supseteq C_N \supseteq C_{N+1} \supseteq \ldots.$$

Proof: Consider an allocation x^* in the core of the $N+1$-replica economy. Suppose it is NOT in the core of the N-replica economy. Then some coalition, S, of the N-replica economy blocks it. But this coalition is also in the $N+1$-replica economy. So S also blocks x^* in the $N+1$-replica economy. This is a contradiction which proves the lemma.

- So we know that the Walrasian equilibria of an economy are contained in the N-replica economies for any N. The set of Walrasian equilibria are the same for each N-replica economy.

- **Theorem:** Core Convergence. If x^* is in the core of every N-replica economy, then x^* is a Walrasian equilibrium. This is the same as saying that as N gets big, the only thing in the core are the Walrasian equilibria. Not proved.

- So we conclude that for large economies, the set of allocations that are Walrasian equilibria is approximately the same as the set of allocations for a voluntary exchange economy.

10 Lecture 10: March 3, 2005

10.1 Social Welfare Theory

- We now consider ways to make policy as a social planner. Given the set of Pareto efficient allocations, which should we choose? By the second welfare theorem, we know that we can arrive at any Pareto efficient allocation, but which one to choose? See G-10.1.

- Consider L goods and I consumers. Preferences are now represented be utility functions:
$$u_i : \Re^L_+ \mapsto \Re.$$
The u_i functions are continuous, strictly increasing and concave. The total endowment, $\overline{\omega} >> 0$.

- Note there is an issue with using utility functions because you may have two utility functions which represent the same preferences that look very different. We won't address this problem here.

- **Definition** Utility Possibility Set (UPS). Consider mapping bundles of goods into utility such that:
$$u_1 = u_1(x_{11}, x_{21}), etc.$$
So define the UPS as:
$$U = \{(u_1, \ldots, u_I) \in \Re^I | \exists \text{ feasible } x \in u_i(x_i) \geq u_i \; \forall \; i\}.$$
See G-10.2. If the individual utility functions are concave, then U is a convex set.

- Example: $u_1 = u_2 = x_1^{1/2} x_2^{1/2}$. $\omega_1 = (1, 0), \omega_2 = (0, 1)$. The Pareto efficient set is found by:
$$Max \; u_1,$$
subject to:
$$u_2 \geq \overline{u}_2, \; x_{11} + x_{12} \leq 1, \; x_{21} + x_{22} \leq 1.$$
This yields a pareto efficient set of:
$$PE = \{((x, x), (1 - x, 1 - x)), 0 \leq x \leq 1\}.$$
Map this set into utilities yields:
$$U = \{(u_1, u_2) = (x, 1 - x), 0 \leq x \leq 1\}.$$

- **Proposition** If an allocation, x^*, is Pareto efficient, then the utility, $u^*(x^*)$, is on the boundry of U.
Proof: Suppose x^* is PE but in the interior of U. Then there is some open ball around x^* with bundles that are strictly better off for all consumers. This means x^* is not Pareto efficient. Hence a contradiction. See G-10.3

- **Definition** Denote the Pareto Frontier as the set: $\{(u_1, \ldots, u_I) \in U$ such that no $(u'_1, \ldots, u'_I)$ with $u'_i \geq u_i \; \forall \; i$ and $u'_i > u_i$ for some $i\}$. This is just the boundry of the UPS.

- See lecture notes for an example with quasi-linear utility. In this case, we have one good, and then money, which can be exchanged so that utility is transfered between consumers on a one-to-one basis. The Pareto frontier is a straight line with slope -1. In general, whenever utility is quasi-linear in one good, you can always just maximize the utility of the good that is not linear and then use the quasi-linear good as a transfer.

- **Definition** Social Welfare Function. $W : U \mapsto \Re$, or $W(u_1, \ldots, u_I) \in \Re$. The W function is a way of determining which allocation we should end up at. How does the social planner value the utility of the consumers? $W(u)$ assigns a "social utility" to each u_i. Again, we have problems here because you cannot in general compare utility between individuals!! But we ignore this!

- So the problem of the social planner:

$$Max_{u \in U} \; W(u).$$

  The solution is the social optimum:

$$u^* = (u_1^*, \ldots, u_I^*).$$

- What forms could $W(u)$ take on? It could be linear:

$$W(u) = \sum \lambda_i u_i = \lambda \cdot u,$$

  where $\lambda_i \geq 0$. This is called the utilitarian (Bentham) social utility function. For $I = 2$, $W = \lambda_1 u_1 + \lambda_2 u_2$. We can define level curves of W as $W(u_1, u_2) =$ constant. For this type of utility, the level curves would be linear with slope $-\lambda_1/\lambda_2$. See G-10.4 and G-10.5.

- So we could rewrite the social planner's problem (SP):

$$Max_x \; \sum_{i=1}^{I} \lambda_i u_i(x_i),$$

  subject to:

$$\sum_i x_i \leq \sum_i \omega_i.$$

- **Proposition** Two points:
  - (1) If $u^* \in U$ solves SP for some vector of weights, $\lambda >> 0$, then u^* is the utility of some Pareto efficient allocation, x^*.

– (2) If U is convex and x^* is any Pareto efficient allocation, then there is a vector of weights, $\lambda \geq 0$ where $\lambda \neq 0$ such that $(u(x_1^*), \ldots, u(x_I^*))$ solves SP.

Thus any Pareto efficient combination of utilities can be achieved with appropriate weights. The second point gives an argument for restricting attention to linear social welfare functions, which are used very often in models. However, with a linear function, the social planner is neutral with respect to inequalities in the distribution of utility. Concave social welfare functions may be used to model preferences for equality.

- Two other types of welfare functions:

$$CES: \; W(u) = \left[\sum_{i=1}^{I} u_i^{\rho} \right]^{1/\rho}, \;\; \rho \in (-\infty, 1), \rho \neq 0.$$

$$Rawlsian - Maximin: \;\; W(u) = Min\{u_1, \ldots, u_i\}.$$

$$Harsanyi: \;\; W(u) = Max_{u \in U} \, \frac{1}{I} \sum_i u_i.$$

So, with CES, we can get varying levels of equality considerations depending on our level of ρ. With the Maximin, we make sure we make the worst guy as well off as possible. And with Harsanyi, we assume each person has an equal chance of becoming one of the I consumers so all consumers want to maximize the expected utility. See G-10.6 for various CES social welfare functions.

Taxation Example

- Consider two consumers and two goods with:

$$u_1 = \sqrt{x_{11}x_{21}}, \; \omega_1 = (1, 0).$$

$$u_2 = \sqrt{x_{22}}, \; \omega_2 = (0, 0).$$

Assume good 1 is a labor/leisure good which consumer 1 has and he can either consume it or make good 2 such that:

$$x_2 = f(x_1).$$

Where $f(\cdot)$ transfers 1 unit of good 1 into 1 unit of good 2.

- The Pareto set is found by:

$$Max \; x_{22}^{1/2}, s.t. \; \sqrt{x_{11}x_{21}} \geq \overline{u}_1, \quad \underbrace{1 - x_{11}}_{Production \; of \; x_2} = x_{21} + x_{22}.$$

Lagrangian:

$$\mathcal{L} = (1 - x_{11} - x_{21})^{1/2} + \lambda[\sqrt{x_{11}x_{21}} - \overline{u}_1].$$

FOCs:

$$\frac{1}{2}(1 - x_{11} - x_{21})^{-1/2} = \frac{1}{2}\lambda x_{11}^{-1/2} x_{21}^{1/2}.$$

$$\frac{1}{2}(1 - x_{11} - x_{21})^{-1/2} = \frac{1}{2}\lambda x_{11}^{1/2} x_{21}^{-1/2}.$$

Thus,

$$x_{11} = x_{21}.$$

So if consumer 1 has all the endowment of good 1 (1 unit) and consumes x of it while using $1 - x$ of it to produce $1 - x$ units of good 2, of which he then consumes x of, it leaves $1 - x - x = 1 - 2x$ of good 2 to be consumed by consumer 2 (as a gift?). Thus the Pareto efficient set is:

$$PE = \{((x, x), (0, 1 - 2x)), 0 \leq x \leq 1/2\}.$$

This translates into utility:

$$U = ((\sqrt{xx} = x), (\sqrt{1 - 2x})).$$

- Now suppose there is a tax, $t \in (0, 1)$ levied on consumer 1's labor. Consumer 1 pays $t(1-x_{11})$ in tax. Assume $p_2 = 1$. Consumer 1 retains $(1-t)(1-x_{11})$ for himself. Thus consumer 1 solves:

$$Max_{x_{11}} \sqrt{x_{11}(1-t)\underbrace{(1 - x_{11})}_{x_2}} = \sqrt{1-t}\sqrt{x_{11}(1 - x_{11})}.$$

FOC:

$$\frac{1}{2}\sqrt{1-t}(x_{11} - x_{11}^2)^{-1/2}(1 - 2x_{11}) = 0.$$

Yields:

$$x_{11} = \frac{1}{2}.$$

So,

$$x_{21} = \frac{1}{2}(1 - t).$$

Thus,

$$U = ((\frac{1}{2}\sqrt{1-t}), (\sqrt{\frac{t}{2}})).$$

See G-10.7 which shows how the frontier shifts in with this distortionary tax.

11 Lecture 11: March 8, 2005

11.1 GE Model of International Trade

Hechsher Ohlin Model of International Trade

- This is a model where countries are assumed to have identical technology and consumers though their relative factor endowments may be different. There are two output goods which are traded between the countries though the two input goods are completely immobile.
- This contrasts Ricardian models where technology is the only thing that is different between the countries.
- So the setup: This is called a $2x2$ production model because we have two countries, A and B, two consumption goods, $q = (q_1, q_2)$, and two input goods or factors, $z = (z_L, z_K)$. Aggregate endowments are thus:

$$\underline{z}^A = (\underline{z}^A_L, \underline{z}^A_K).$$

$$\underline{z}^B = (\underline{z}^B_L, \underline{z}^B_K).$$

Initially, countries have no units of the output goods and only have endowments of the inputs.

- Since technology is identical across countries, assume each country has two firms with technology:

$$\text{Firm 1: } q_1 = f_1(z_{L1}, z_{K1}),$$

$$\text{Firm 2: } q_2 = f_2(z_{L2}, z_{K2}).$$

Then $z = ((z_{L1}, z_{K1}), (z_{L2}, z_{K2}))$ is feasible for country i if:

$$z^*_{L1} + z^*_{L2} \leq \underline{z}^i_L.$$

$$z^*_{K1} + z^*_{K2} \leq \underline{z}^i_K.$$

We also assume that the production functions are C^1, strictly increasing, concave and exhibit Constant Returns to Scale. CRS is a strong assumption. Two such production functions might be:

$$f(z_L, z_K) = z_L^{\alpha} z_K^{1-\alpha}, \quad \alpha \in (0,1).$$

$$f(z_L, z_K) = (z_L^{\rho} + z_K^{\rho})^{1/\rho}, \quad \rho \in (0,1).$$

Firms are price takes with the price vector for each country:

$$(p_1, p_2, w_L, w_K),$$

as the output and inputs prices.

- Consumers. Assume there are I consumers who are price takers and who consume ONLY output goods. They also own shares of the firms and the inputs to production.

Since they don't value these inputs, we assume they sell all of them to the firms at the going wages. We'll see how consumers enter the model next lecture.

- So the two countries trade the output good only. At equilibrium, standard theory tells us that the price of the output goods must be the same in both countries. We are interested in the prices of the non-tradable input goods. What do we know?

- Consider the $2x2$ production model in just one country. Firms take prices as given. We can define the Pareto set of factor allocations as:

$$Max_{z_1}\ f_1(z_{L1}, z_{K1}),$$

such that:

$$f_2(z_{L2}, z_{K2}) \geq \underline{q}_2.$$
$$z_{L1} + z_{L2} \leq \underline{z}_L.$$
$$z_{K1} + z_{K2} \leq \underline{z}_K.$$

This is the same way we traced out the Pareto set for utility of consumption. Maximize one's production subject to the other attaining some minimum level of output. Our lagrangian is then:

$$\mathcal{L} = f_1(z_{L1}, z_{K1}) + \lambda[\underline{q}_2 - f_2(\underline{z}_L - z_{L1}, \underline{z}_K - z_{K1})].$$

FOCs yields the familiar result:

$$\frac{\partial f_1/\partial z_{L1}}{\partial f_1/\partial z_{K1}} = \frac{\partial f_2/\partial z_{L2}}{\partial f_2/\partial z_{K2}}.$$

Or,

$$MRTS_1 = MRTS_2.$$

- See G-11.1. The tangencies between the $MRTS$'s trace out the Pareto set in the edgeworth box. Note this set is just Pareto optimal with respect to the division of two inputs between two firms, not the entire economy. Note we have isoquants instead of indifference curves in our box now.

- So taking the output price vector, $p = (p_1, p_2)$, as given, an input price vector, $(w_L, w_K) >> 0$, is a factor market equilibrium price if firm's demand for labor and capital equal the total endowment of the factors in the country.

- **Definition:** An Interior Factor Market Equilibrium. For output prices, $p = (p_1, p_2)$, an interior factor market equilibrium is a price vector $w^* >> 0$ and an allocation, $z^* = ((z^*_{L1}, z^*_{K1}), (z^*_{L2}, z^*_{K2}))$, such that:
  - (1) Firm's maximize profits with inputs (z^*_{Li}, z^*_{Ki}) such that $q^*_i = f_i(z^*_{Li}, z^*_{Ki}) > 0$ for $i = 1, 2$. So each firm must produce a positive quantity.
  - (2) Input market's clear:

$$z^*_{L1} + z^*_{L2} = \underline{z}_L.$$

$$z^*_{K1} + z^*_{K2} = \underline{z}_K.$$

- So there are two ways we can calculate the profit maximization problem of the firm: using the production function or using the cost function.

- Consider firm the profit maximization problem of firm j using its production function:

$$Max_{z_j}\ [p_j f_j(z_{Lj}, z_{Kj}) - w^*_L z_{Lj} - w^*_K z_{Kj}].$$

  The FOC implies:

$$\frac{\partial f_j/\partial z_{Lj}}{\partial f_j/\partial z_{Kj}} = \frac{w^*_L}{w^*_K}.$$

  So we have the general result:

$$MRTS_1 = MRTS_2 = \frac{w^*_L}{w^*_K}.$$

- Now consider maximizing profits using a cost function. For an output level of q_j, the cost of production would be:

$$C_j(w, q_j) = Min\ (w_L z_{Lj} + w_K z_{Kj}),$$

  such that,

$$f_j(z_{Lj}, z_{Kj}) = q_j.$$

  Solving this yields conditional factor demands: $z_{Lj}(w^*_L, w^*_K; q_j)$ and $z_{Kj}(w^*_L, w^*_K; q_j)$. Then, consider the problem of the firm:

$$Max_{q_j}\ [p_j q_j - C_j(w, q_j)].$$

  FOC:

$$p_j = \frac{\partial C_j}{\partial q_j},$$

  or price = marginal cost!

- So we can reformulate our equilibrium as: given output prices, $p = (p_1, p_2)$, an interior factor market equilibrium is a vector of input prices, $(w^*_L, w^*_K) >> 0$ and output levels, $q^* = (q^*_1, q^*_2)$, such that:

  - Firms maximize profits which implies:

$$p_j = \frac{\partial C_j(w^*_L, w^*_K, q^*_j)}{\partial q_j},\ j = 1, 2.$$

  - Markets clear:

$$z^*_{L1}(w^*_L, w^*_K; q^*_1) + z^*_{L2}(w^*_L, w^*_K; q^*_2) = \underline{z}_L.$$
$$z^*_{K1}(w^*_L, w^*_K; q^*_1) + z^*_{K2}(w^*_L, w^*_K; q^*_2) = \underline{z}_K.$$

- Recall from consumer and producer theory, Shepard's lemma:

$$z_{Lj}(w_L, w_K; q_j) = \frac{\partial C_j(w_L, w_K; q_j)}{\partial w_L}.$$

$$z_{Kj}(w_L, w_K; q_j) = \frac{\partial C_j(w_L, w_K; q_j)}{\partial w_K}.$$

Also, since we have assumed a CRS production function, it is clear that:

$$C_j(w_L, w_K; q_j) = q_j c_j(w_L, w_K),$$

where c_j is the marginal cost of producing good j. Thus, at an interior factor market equilibrium, the profit maximizing condition is:

$$p_1 = c_1(w_L^*, w_K^*), \text{and, } p_2 = c_2(w_L^*, w_K^*).$$

- So we have two equations and two unknowns. Is this solvable for a unique solution? Possibly. If for every $p = (p_1, p_2)$, these equations did have a unique (w_L^*, w_K^*) then the input prices will NOT depend on the aggregate endowments of labor and capital! This leads to something called the Factor Price Equilization. Trade in the output good alone is enough to induce input prices to equalize even though factors of production are assumed to be immobile. So country A could have most of the capital and country B could have most of the labor, the prices of labor will end up being the same in both countries and the same for capital. Key result in the trade literature.

- So how do we know if have a unique equilibrium? Consider G-11.2. In the input price space, these two curves represent the profit maximizing condition (price equals marginal cost), for various levels of the input prices. If we can show that one curve is always steeper than the other, this would imply that there could be no more than one solution (though I think there still might be zero).

- So consider the slope of the first curve. Write:

$$\Psi = c_1(w_L^*, w_K^*) - p_1 = 0.$$

Then, by the IFT,

$$\frac{\partial w_K}{\partial w_L} = -\frac{\partial \Psi / \partial w_L}{\partial \Psi / \partial w_K}.$$

Or,

$$\frac{\partial w_K}{\partial w_L} = -\frac{\partial c_1(w_L, w_K)/\partial w_L}{\partial c_1(w_L, w_K)/\partial w_K}.$$

Similarly for the second curve:

$$\frac{\partial w_K}{\partial w_L} = -\frac{\partial c_2(w_L, w_K)/\partial w_L}{\partial c_2(w_L, w_K)/\partial w_K}.$$

Via shepards, lemma, (note $c_j(w_L, w_K) = C_j(w_L, w_K, 1)$),

$$\text{Good 1 Curve: } \frac{\partial w_K}{\partial w_L} = -\frac{z_{L1}(w_L, w_K, 1)}{z_{K1}(w_L, w_K, 1)}.$$

$$\text{Good 2 Curve: } \frac{\partial w_K}{\partial w_L} = -\frac{z_{L2}(w_L, w_K, 1)}{z_{K2}(w_L, w_K, 1)}.$$

Let $a_{Lj}(w_L, w_K) = z_{Lj}(w_L, w_K, 1)$ and $a_{Kj}(w_L, w_K) = z_{Kj}(w_L, w_K, 1)$ for $j = 1, 2$. Thus we have:

$$\text{Good 1 Curve: } \frac{\partial w_K}{\partial w_L} = -\frac{a_{L1}(w_L, w_K)}{a_{K1}(w_L, w_K)} \equiv \text{Factor Intensity of Firm 1.}$$

$$\text{Good 2 Curve: } \frac{\partial w_K}{\partial w_L} = -\frac{a_{L2}(w_L, w_K)}{a_{K2}(w_L, w_K)} \equiv \text{Factor Intensity of Firm 2.}$$

- So we have a result: Factor Intensity Assumption. If:

$$\frac{a_{L1}(w_L, w_K)}{a_{K1}(w_L, w_K)} > \frac{a_{L2}(w_L, w_K)}{a_{K2}(w_L, w_K)}, \ \forall\ (w_L, w_K),$$

then the two equations have at MOST one solution, (w_L^*, w_K^*), so we can prove the factor equilization theorem.

11.2 Completely Worked Example

- Consider two firms with technologies:

$$f_1 = Z_{L1}^{2/3} Z_{K1}^{1/3},$$

$$f_2 = Z_{L1}^{1/3} Z_{K1}^{2/3}.$$

Endowment:

$$\underline{Z} = (\underline{Z}_L, \underline{Z}_K) = (1, 1).$$

- The Pareto set is characterized by the condition, $MRTS_1 = MRTS_2$:

$$\frac{\partial f_1/\partial z_{L1}}{\partial f_1/\partial z_{K1}} = \frac{\partial f_2/\partial z_{L2}}{\partial f_2/\partial z_{K2}}.$$

Or,

$$2(Z_{K1}/Z_{L1}) = \frac{1}{2}(Z_{K2}/Z_{L2}).$$

Solving,

$$Z_{K1} = \frac{Z_{L1}}{4 - 3Z_{L1}}.$$

They Pareto set is shown in G-11.3. Notice that it lies below the 45.

- Now, lets do profit maximization with the cost function. First:

$$C_1(w, q_1) = Min\ \{w_L Z_{L1} + w_K Z_{K1}\},$$

subject to:

$$Z_{L1}^{2/3} Z_{K1}^{1/3} = q_1.$$

Yields:

$$2(Z_{K1}/Z_{L1}) = w_L/w_K.$$

Plugging this into the constraint yields:

$$Z_{L1} = q_1(\frac{2w_K}{w_L})^{1/3}.$$

$$Z_{K1} = q_1(\frac{w_L}{2w_K})^{2/3}.$$

These are our conditional factor demands. Put them back into our cost function yields:

$$C_1(w_L, w_K, q_1) = q_1 w_L^{2/3} w_K^{1/3}(2^{1/3} + 2^{-2/3}).$$

And similarly for firm 2:

$$C_2(w_L, w_K, q_2) = q_2 w_L^{1/3} w_K^{2/3}(2^{1/3} + 2^{-2/3}).$$

- And now use the cost functions to do profit maximization:

$$Max\ p_j q_j - C_j.$$

Yields price equals marginal cost or:

$$p_1 = w_L^{2/3} w_K^{1/3}(2^{1/3} + 2^{-2/3}).$$

$$p_2 = w_L^{1/3} w_K^{2/3}(2^{1/3} + 2^{-2/3}).$$

- And these are our two equations and two unknowns. Evaluating our conditional factor demands at $q_j = 1$, yeilds:

$$a_{L1}(w_L, w_K) = (\frac{2w_K}{w_L})^{1/3}.$$

$$a_{K1}(w_L, w_K) = (\frac{w_K}{2w_L})^{2/3}.$$

Thus, the factor intensity of each firm becomes:

$$\frac{a_{L1}}{a_{K1}} = 2\frac{w_K}{w_L}.$$

$$\frac{a_{L2}}{a_{K2}} = \frac{1}{2}\frac{w_K}{w_L}.$$

Thus, firm one is MORE labor intensive (as we knew all along) so:

$$\frac{a_{L1}}{a_{K1}} > \frac{a_{L2}}{a_{K2}}, \ \forall \, (w_L, w_K).$$

Thus, if there is a solution, it is UNIQUE!

- Given international prices and factor endowments, we could set $price = MC$ and solve for equilibrium wages, and then use the resource constraint to solve for q^* and z^*.

12 Lecture 12: March 10, 2005

12.1 GE Model of International Trade II

Hecksher Ohlin Model of International Trade

- We continue the GE model of trade.
- Recall we have solved for the cost function for each firm and set the marginal cost equal to the price to solve for the equilibrium wages. Note if we can show that the level sets of the $P - MC$ curves have different slopes, we might be able to show that they cross just once which shows that there is a unique equilibrium, *no matter what the initial factor endowments are!!*
- So we look at relative factor intensities and if one firm is more intensive in, say labor, then there can be at MOST one crossing. There still may not be an equilibrium. However, if it exists, it is unique.
- **Theorem:** Factor Price Equalization. Suppose that one factor is used more intensively in one firm than another. Given international prices, (p_1, p_2), if each country is at an interior factor market equilibrium (ie, the levels sets cross once in the positive quadrant), then the domestic factor prices, w_L^* and w_K^* are the same in both countries. So trade in the output goods equalizes the prices of the inputs. See G-12.1.
- **Theorem:** Stolper-Samuelson Theorem. In the $2x2$ production economy with the factor intensity assumption, if the price of good 1 increases, then the price of the input (say labor) that is intensive in that good, will increase. See G-12.2. Similarly, if the price of good 2 increases, the factor that is used intensively in production of good 2 will also increase.
- **Theorem:** Rybcszynski Theorem. In a $2x2$ production economy with the factor intensity assumption, if a country's endowment of a factor increases (say labor), then it will produce more of the good that uses labor intensively in production, and less of the other good. There is an edgeworth box analysis for this in MWG. You could also take the resource constraints (with general endowments) and solve for q_1 and q_2 and then show what happens to:
$$\frac{\partial q_1}{\partial \bar{z}_L}.$$
- So far we have only considered the firms in the economy. What about the consumers? We need them to make this a true GE model. This is where Hecksher-Ohlin comes in.
- Hecksher-Ohlin Assumptions.
  - (1) 2 countries, 2x2 production, same CRS technology for firms in each country.
  - (2) Factor intensity assumption: good 1 is more labor intensive than good 2.

- (3) Each country has the same representative consumer with a continuous, strictly concave, strictly increasing, homothetic utility function over the two consumption goods.
- (4) Country A is better endowed with labor than capital relative to B.

- A free trade equilibrium consists of a price vector:

$$(p_1^*, p_2^*, w_{LA}^*, w_{KA}^*, w_{LB}^*, w_{KB}^*),$$

and an allocation of goods to consumers and firms such that:

  - (1) Each country is at a factor market equilibrium given input and output prices.
  - (2) The representative consumer in each country is maximizing utility subject to his budget constraint.
  - (3) International markets clear.

- **Theorem:** Hecksher-Ohlin. Given the above assumptions, if there is a free-trade equilibrium in which neither country specializes, $(q_{1A}, q_{2A}, q_{1B}, q_{2B} > 0)$, then the factor prices are equalized and each country exports the good whose production is relatively more intensive in the input factor with which the country is relatively better endowed. So if you have lots of machines – you export the good that uses the machines.

Midterm Review

12.2 Important Notes and Theorems

- In an exchange economy, an allocation and price vector are a WE if firms maximize profits, consumers maximize utility and markets clear.

- Pareto Efficiency: A feasible allocation (x, y) such that no other feasible allocation (x', y') such that $x_i' \succeq_i x_i \; \forall \; i$ and $x_i' \succ x_i$ for some i.

- **Theorem:** First Welfare Theorem: All WE are PE. – Required assumption: LNS.

- **Theorem:** Second Welfare Theorem: Any PE allocation can be attained by using an appropriate transfer of wealth with a decentralized market system. – Required assumption: convex, continuous, strictly monotonic preferences; production set is convex; economy has a strictly positive production vector. So we can allocation wealth in such a way to let the market work and send us towards any PE point. Wealth redistribution must be possible.

- Existence - Preferences are continuous, strictly convex and strongly monotone. Given this, we have the four properties of $Z(p)$ [for all $p >> 0$]:

  - (1) Continuous.
  - (2) $Z(\lambda p) = Z(p)$.
  - (3) $p \cdot Z(p) = 0$.
  - (4) If $p^n \to \overline{p} \neq 0$ where $\overline{p}_l = 0$ for some l, then $Z_l(p) \to \infty$, for SOME l.

  Thus we can say that if the 4 assumptions hold, we have the correct preference assumptions and there is at least one WE. If assumptions hold, WE solves $Z(p) = 0$.

- Uniqueness. Assume preferences are continuous, strictly convex and strongly monotone. Might get multiple equilibria if wealth effects are strong enough.

- Let $\phi = D\hat{Z}(p)$ be the matrix of price effects for the first $L-1$ goods. If ϕ is invertible (full rank or $|\phi| = 0$), then p is regular. Regular price vectors are locally unique. A regular economy has a finite number of equilibrium price vectors. The set of all endowments which induce a regular economy is generic. Thus economies with a continuum of WE are not-generic and have measure zero.

- Global uniqueness: gross substitutes property or the weak axiom. Still need continuous, strictly convex, strongly monotone prefs with $Z(p)$ satisfying A1-A4. Economy has GS property if:

  $$\frac{\partial Z_{\tilde{l}}}{\partial p_l} > 0 \; \forall \; l \neq \tilde{l}.$$

  So ϕ must have negative diagonal entries and positive off diagonal entries. GS property is additive over consumers.

- Global uniqueness with GS: If $Z(p)$ satisfies A1-A4 and the GS property then the economy has one normalized equilibrium price vector.
- Weak axiom. for any pair, p,p':

$$Z(p) \neq Z(p') \; and \; p \cdot Z(p') \leq 0 \Longrightarrow p' \cdot Z(p) > 0.$$

- Global uniqueness with WA: If $Z(p)$ satisfies A1-A4 and the WA and the economy is regular, then the economy has one normalized equilibrium price vector.
- Note that the WA is always satsified for an individual but it is NOT additive – imposing in the aggregate is damn strong.
- Stability: $\dot{p}_l = Z_l(p)$. $Z_l(p) = 0$ at equilibrium.
- Global Stability: Existence + Uniquness = Global stability.
- To find core, consider coalitions which can block possibly PE allocations. Core is all feasible allocations that are NOT blocked.
- Core allocations are PE. $WE \subseteq CORE \subseteq PE$.
- Equal treatment property: If an allocation,x is in the core of an N-rep economy, all consumers of the same type have the same bundle.
- Shrinking Core Lemma: The core of an N-rep contains the core of an N+1-rep.
- Core convergence: If x^* is in the core of every N rep economy, then x^* is a WE.
- UPS:

$$U = \{(u_1, \ldots, u_I) \in \Re^I \ni \; \exists \, feasible \; x \ni u_i \leq u_i(x_i) \; \forall i\}.$$

- PE allocations are on the boundry of the UPS – the so called Pareto frontier.
- If U is convex, and x^* is on the frontier, then there is a vector of weights, λ, such that $u(x^*)$ solves the SP, $Max \; U = \sum_i \lambda_i u_i$. However, linear social welfare functions are neutral with respect to equality. Need concave SWFs to get equality issues.
- **Definition:** An Interior Factor Market Equilibrium. For output prices, $p = (p_1, p_2)$, an interior factor market equilibrium is a price vector $w^* >> 0$ and an allocation, $z^* = ((z^*_{L1}, z^*_{K1}), (z^*_{L2}, z^*_{K2}))$, such that:
  - (1) Firm's maximize profits with inputs (z^*_{Li}, z^*_{Ki}) such that $q^*_i = f_i(z^*_{Li}, z^*_{Ki}) > 0$ for $i = 1, 2$. So each firm must produce a positive quantity.

$$p_j = \frac{\partial C_j(w^*_L, w^*_K, q^*_j)}{\partial q_j}, \; j = 1, 2.$$

– (2) Input market's clear:

$$z^*_{L1} + z^*_{L2} = \underline{z}_L.$$

$$z^*_{K1} + z^*_{K2} = \underline{z}_K.$$

- Recall from consumer and producer theory, Shepard's lemma:

$$z_{Lj}(w_L, w_K; q_j) = \frac{\partial C_j(w_L, w_K; q_j)}{\partial w_L}.$$

$$z_{Kj}(w_L, w_K; q_j) = \frac{\partial C_j(w_L, w_K; q_j)}{\partial w_K}.$$

- Factor Intensity Assumption. If:

$$\frac{a_{L1}(w_L, w_K)}{a_{K1}(w_L, w_K)} > \frac{a_{L2}(w_L, w_K)}{a_{K2}(w_L, w_K)},\ \forall\,(w_L, w_K),$$

then the two equations have at MOST one solution, (w^*_L, w^*_K), so we can prove the factor equilization theorem.

- **Theorem:** Factor Price Equalization. Suppose that one factor is used more intensively in one firm than another. Given international prices, (p_1, p_2), if each country is at an interior factor market equilibrium (ie, the levels sets cross once in the positive quadrant), then the domestic factor prices, w^*_L and w^*_K are the same in both countries. So trade in the output goods equalizes the prices of the inputs.

- **Theorem:** Stolper-Samuelson Theorem. In the $2x2$ production economy with the factor intensity assumption, if the price of good 1 increases, then the price of the input (say labor) that is intensive in that good, will increase. Similarly, if the price of good 2 increases, the factor that is used intensively in production of good 2 will also increase.

- **Theorem:** Rybcszynski Theorem. In a $2x2$ production economy with the factor intensity assumption, if a country's endowment of a factor increases (say labor), then it will produce more of the good that uses labor intensively in production, and less of the other good. There is an edgeworth box analysis for this in MWG. You could also take the resource constraints (with general endowments) and solve for q_1 and q_2 and then show what happens to:

$$\frac{\partial q_1}{\partial \overline{z}_L}.$$

- Hecksher-Ohlin Assumptions.
  - (1) 2 countries, 2x2 production, same CRS technology for firms in each country.
  - (2) Factor intensity assumption: good 1 is more labor intensive than good 2.
  - (3) Each country has the same representative consumer with a continuous, strictly concave, strictly increasing, homothetic utility function over the two consumption goods.

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- A free trade equilibrium consists of a price vector:

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  and an allocation of goods to consumers and firms such that:

  - (1) Each country is at a factor market equilibrium given input and output prices.
  - (2) The representative consumer in each country is maximizing utility subject to his budget constraint.
  - (3) International markets clear.

- **Theorem:** Hecksher-Ohlin. Given the above assumptions, if there is a free-trade equilibrium in which neither country specializes, $(q_{1A}, q_{2A}, q_{1B}, q_{2B} > 0)$, then the factor prices are equalized and each country exports the good whose production is relatively more intensive in the input factor with which the country is relatively better endowed. So if you have lots of machines – you export the good that uses the machines.

12.3 Notes from Problem Sets and Exams

- To find the pareto set, either max one's utility holding the other to be at least some level, or totally differentiate $u_1 = \overline{u}$ and set MRSs equal. Add feasibility to get pareto set.

- Check over proof of A4 for $L = 2$.

- For CES utility, if $\rho \in (0, 1)$, the GS property holds. For CD utility, need $\sum \alpha_i = 1$.

- Scarf: even if the economy has a unique WE, it may not be reached if the dynamics are just right. Could get a circular pattern.

- When defining the pareto set or the UPS, remember to give limits on the allocations or levels of utility.

- Concave social welfare functions $\Rightarrow$ "aversion to inequality".

- Curves in Stolper-Samuelson ($p = mc$) are isocost curves.

Begin Kranton Lectures

13 Lecture 13: March 10, 2005

13.1 Topic 1: Uncertainty and Utility

Formalization of Uncertainty

- We want to consider how individuals make decisions when the consequences are uncertain.
- Let C denote the set of all possible outcomes, finite in number. Thus,

$$C = \{c_1, \ldots, c_N\}.$$

- **Definition:** A simple lottery, L, is a vector of probabilities:

$$L = (p_1, \ldots, p_N),$$

  where $p_n = Prob(c_n)$. So $p_n \geq 0 \; \forall \; n = 1, \ldots, N$ and $\sum_n p_n = 1$.

- **Definition:** A compound lottery, L^c. Given K simple lotteries, $(L_1, \ldots, L_K)$, where $L_k = (p_1^k, \ldots, p_N^k)$, a compound lottery is:

$$L^c = (L_1, \ldots, L_K; \alpha_1, \ldots, \alpha_K),$$

  which yields the "simple lottery" L_k with probability α_k. Of course $\alpha_k \geq 0 \; \forall \; k = 1, \ldots, K$ and $\sum_k \alpha_k = 1$.

- **Definition:** Reduced simple lottery. $L = (p_1, \ldots, p_N)$ gives the same probability of outcomes as L^c so:

$$p_n = \alpha_1 p_n^1 + \cdots + \alpha_K p_n^K.$$

  So we are just combining all the simple lotteries.

Preferences over Lotteries

- Let C be the set of all possible outcomes and $\mathcal{L}$ be the set of all simple lotteries over C.
- Let $\succeq$ be the preference relation over elements of $\mathcal{L}$.
- We seek to develop reasonable axioms about an individual's preference relation and then show that these preferences can be represented with a utility function:

$$U : \mathcal{L} \mapsto \Re.$$

We want U to be continuous and to reflect accurately the preference relation, ie:

$$L \succeq L' \text{ iff } U(L) \geq U(L').$$

- The Axioms:
  - (A1) Completeness. For any $L, L' \in \mathcal{L}$, either $L \succeq L'$, $L' \succeq L$, or both.
  - (A2) Transitivity. For any $L, L', L'' \in \mathcal{L}$, if $L \succeq L'$ and $L' \succeq L''$, then $L \succeq L''$.
  - (A3) Continuity. For any $L, L', L'' \in \mathcal{L}$, such that $L \succ L' \succ L''$, there exists $\alpha \in (0, 1)$ such that:
  $$\alpha L + (1 - \alpha)L'' \sim L'.$$
  This axiom rules out lexicographic preferences. One lottery cannot strictly dominate any others.
  - (A4) Independence. For all $L, L', L'' \in \mathcal{L}$ and $\alpha \in (0, 1)$,
  $$L \succ L' \text{ iff } \alpha L + (1 - \alpha)L'' \succ \alpha L' + (1 - \alpha)L''.$$
  This is like the independence of irrelevant alternatives.

- The continuity axiom impiles the existence of a continuous utility function such that:
$$U : \mathcal{L} \mapsto \Re, \ s.t. \ U(L) > U(L') \text{ iff } L \succ L'.$$
$$U : \mathcal{L} \mapsto \Re, \ s.t. \ U(L) = U(L') \text{ iff } L \sim L'.$$

- The independence axiom gives us a utility function that has an expected utility representation of the preference relation.

- **Definition:** A utility function has an expected utility form if there is an assignment of numbers, $(u_1, \ldots, u_N)$, to the N outcomes such that for every simply lottery, $L \in \mathcal{L}$, we have:
$$U(L) = u_1 p_1 + \cdots + u_N p_N.$$
This is called a von Neumann-Morganstern expected utility function (VNM).

- **Proposition** The expected utility theorem: Suppose $\succeq$ satisfies the axioms above. Then there exists a utility representation of $\succeq$ that has the expected utility form. That is, we can assign numbers to the N outcomes such that for any two lotteries, L and L', we have $L \succ L'$ iff:
$$\sum_{n=1}^{N} u_n p_n > \sum_{n=1}^{N} u_n p'_n.$$

- Note that U is called the VNM utility function and little u is often referred to as the Bernoulli utility function.

- **Proposition** Suppose $U : \mathcal{L} \mapsto \Re$ is a VNM utility function representing $\succeq$. Then $\tilde{U}$ is another VNM utility function representing $\succeq$ if and only if:

$$\tilde{U}(L) = \beta U(L) + \gamma,\ \beta > 0,\ \forall\ L \in \mathcal{L}.$$

Thus utility functions are only equivalent through affine tranformations (NOT monotonic tranformations as before).

- Example. We will show that the previous proposition holds for two VNM utility functions. Suppose U and $\tilde{U} : \mathcal{L} \mapsto \Re$ both represent $\succeq$. Suppose:

$$\tilde{U}(L) = [U(L)]^2.$$

Consider two lotteries, L and L', such that L puts probability one on outcome i and L' puts probability one on outcome j. Suppose $L \succ L'$. Then:

$$U(L) = u_i > u_j = U(L').$$

Since both U and $\tilde{U}$ have the expected utility form:

$$\tilde{U}(L) = \tilde{u}_i = (u_i)^2,$$

$$\tilde{U}(L') = \tilde{u}_j = (u_j)^2.$$

Since there is only one term that has positive probability, we can do this. Now consider a third lottery L'' that puts equal probability on both u_i and u_j. Thus,

$$\tilde{U}(L'') = \frac{1}{2}\tilde{u}_i + \frac{1}{2}\tilde{u}_j = \left(\frac{1}{2}u_i + \frac{1}{2}u_j\right)^2 = [U(L'')]^2.$$

Substituting on the LHS yields:

$$\frac{1}{2}(u_i)^2 + \frac{1}{2}(u_j)^2 = \left(\frac{1}{2}u_i + \frac{1}{2}u_j\right)^2 \Longrightarrow u_i = u_j.$$

Since this is a contradiction to our original assumption, U and $\tilde{U}$ cannot represent the same preferences.

14 Lecture 14: April 5, 2005

14.1 Topic 1: Uncertainty and Utility

- Recall from last time we had our VNM utility function, $U : \mathcal{L} \mapsto \Re$ and our Bernoulli utility function, u : outcomes $\mapsto \Re$. Then $\tilde{U}$ represents the same preferences as U if $\tilde{U}$ is an affine transform of U.
- We had 4 axioms that U satisfied: completeness, transitivity, continuity, and independence.

Problems with the Theory

- Allais Paradox. Consider dollar outcomes: $C = \{4000, 3000, 0\}$. Suppose we have scenario A:

  $$L_A = (0.8, 0.0, 0.2) \text{ or } L'_A = (0.0, 1.0, 0.0).$$

  And scenario B:

  $$L_B = (0.2, 0.0, 0.8) \text{ or } L'_B = (0.0, 0.25, 0.75).$$

  Most people prefer $L'_A \succ L_A$ and $L_B \succ L'_B$. Can we construct a utility function such that:

  $$U(L'_A) > U(L_A) \text{ and } U(L_B) > U(L'_B).$$

  No because, given $u(0) = 0$,

  $$U(L_A) = 0.8 * u(4000) + 0.0 * u(3000) + 0.2 * u(0) = 0.8 * u(4000).$$

  $$U(L'_A) = u(3000).$$

  Thus $U(L'_A) \succ U(L_A)$ iff $u(3000) > 0.8 * u(4000)$. Or, $0.25 * u(3000) > 0.2 * u(4000) \Longrightarrow L'_B \succ L_B$. A contradiction!
- What is violated in the Allais paradox? The independence axiom. Consider $L'' = (0, 0, 1)$. Thus if $L'_A \succ L_A$, then (by independence):

  $$0.25L'_A + 0.75L'' \succ 0.25L_A + 0.75L''.$$

  Or,

  $$0.25 * (0, 1, 0) + 0.75 * (0, 0, 1) \succ 0.25 * (0.8, 0, 0.2) + 0.75 * (0, 0, 1)$$

  $$\Rightarrow (0, 0.25, 0.75) \succ (0.2, 0, 0.8) \Rightarrow L'_B \succ L_B.$$
- Another problem with expected utility theory is Framing. In Kahneman and Tversky, they offer people two alternatives: surgery and radiation to treat their cancer. If the statistics are presented in a survival fashion (as in, your chances of surviving are ...) then people are more willing to forgo a higher chance of dying today given their long term chances are better. If the statistics are presented in a mortality fashion (as in, your chances of dying are ...), then people prefer to make sure they don't die today even if their long term chances are worse. Thus the framing of the question is important.

- Another example: a customer is buying a calculator and a stereo at a store and finds out that one of the items is available for five dollars less elsewhere. When the discounted item is the calculator, the customer is more willing to make the trip, while when the savings are a smaller percentage of the price, they are more likely to buy from the current store.

- It is possible to frame the question in a neutral fashion. In calculator example, suppose the alternative store offered a five dollar rebate on any item purchased. This would eliminate the framing problem.

14.2 Topic 2: Risk Aversion and Expected Utility

Uncertainty over Monetary Outcomes

- Suppose x is a monetary outcome. $C = [a, b] \subseteq \Re$. A lottery L is a CDF, $F : \Re \mapsto [0, 1]$, not necessarily continuous. Denote $f(x)$ the pdf of x. Then the expected value of the lottery is:
$$E[L] = \int_{-\infty}^{\infty} x f(x) dx.$$
Note this could be discrete as well.

- Thus by the expected utility theorem, the expect utility of a lottery, F is:
$$U(F) = \int_{-\infty}^{\infty} u(x) f(x) dx.$$
Remember $U(F)$ is the VNM utility function and $u(x)$ is the bernoulli utility function.

- Example: St. Petersburg Game. Suppose we have a game where one person flips a coin repeatedly and you win 2^n dollars where n is the number of flips required to get the first head. Then,
$$EV[L] = 0.5 * 2 + 0.5 * 0.5 * 2^2 + 0.5^2 * 0.5 * 2^3 + \cdots = \infty.$$
However, people are only usually willing to pay five or six dollars to play this game! The reason is that we are not discounting large values (unlikely values) enough. Since the payoff grows so quickly, this exactly offsets the decreasing probability of not throwing a head. So we might need a concave utility function.

- **Definition** Weakly Risk Averse. An agent is (weakly) risk averse if for any lottery, $F(\cdot)$, the degenerate lottery that places probability one on the mean of F is (weakly) preferred to the lottery, F. If the individual is always indifferent between these two lotteries, then we say that individual is risk neutral. An individual is a risk lover if a degenerate lottery is never preferred to the lottery F.

- Thus, an agent is therefore risk averse iff:

$$\int_{-\infty}^{\infty} u(x)f(x)dx \leq u\left[\int_{-\infty}^{\infty} xf(x)dx\right] \; \forall \; F(\cdot).$$

See G-14.1. This last expression is Jensen's inequality which defines a concave function. Thus we have the following implications regarding our Bernoulli utility function, $u(\cdot)$:

$$\text{Strict Concavity} \Longleftrightarrow \text{Strict Risk Aversion.}$$

$$\text{Linearity} \Longleftrightarrow \text{Risk Neutrality.}$$

$$\text{Strict Convexity} \Longleftrightarrow \text{Strict Risk Loving.}$$

- **Definition** Certainty Equivalence. The CE of a lottery, $F(\cdot)$, denoted, $c(F, u)$, is the quantity that satisifes:

$$U(F'') = u(c(F, u)) = \int_{-\infty}^{\infty} u(x)f(x)dx = U(F).$$

So it is a number such that if the agent gets this number with probability one, he is indifferent between this and the expected utility of the lottery. See G-14.2.

- **Definition** Risk Premium. Given a Bernoulli utility function, $u(\cdot)$, and a lottery, $F(\cdot)$, the risk premium, denoted, $\rho(F, u)$, is the difference between the mean of F and the certainty equivalence, $c(F, u)$:

$$\rho(F, u) = \int_{-\infty}^{\infty} xf(x)dx - c(F, u) = EV - CE.$$

Thus, the risk premium is the maximum amount an agent would be willing to pay to "get rid of the risk."

15 Lecture 16: April 7, 2005

15.1 Topic 2 continued

- **Definition** Probability premium, denoted $\pi(x, \epsilon, u)$, solves the following:

$$u(x) = (\frac{1}{2} + \pi(x, \epsilon, u))u(x + \epsilon) + (\frac{1}{2} - \pi(x, \epsilon, u))u(x - \epsilon).$$

So it's the additional probability that must be placed on the better of two outcomes in order for a person to be indifferent between that lottery and a lottery which places equal probability on both outcomes. So we can get x for sure, or we can face uncertainty and get a little more than x or a little less but these won't be equally likely to get indifference. We need assurance that the better outcome happens more often. See G-15-1.

Application - Risk Aversion and Insurance

- Consider a risk adverse agent with initial wealth, w. The agent faces a loss of D with probability π. The agent can buy insurance at q\$ per unit which pays 1\$ per unit when the loss occurs.

- The problem of the agent is to determine how much insurance (α) to purchase. The outcome space is:

$$C = \{\underbrace{w - \alpha q}_{no\ loss}, \underbrace{w - \alpha q - D + \alpha}_{loss}\}.$$

The lottery is defined as:

$$L = (1 - \pi, \pi).$$

- Agent's expected wealth:

$$E[W] = w - \alpha q - \pi(D - \alpha).$$

- Maximization:

$$Max_{\alpha} \{EU = (1 - \pi)u(w - \alpha q) + \pi u(w + (1 - q)\alpha - D)\}.$$

FOC(α):

$$\underbrace{-q(1 - \pi)u'(w - \alpha^* q)}_{MC\ of\ insurance} + \underbrace{(1 - q)\pi u'(w + (1 - q)\alpha^* - D)}_{MB\ of\ insurance} = 0.$$

- Now assuming that profits of the insurance firm are: $\Pi = \alpha q - \pi\alpha, \Pi = 0 \Rightarrow q = \pi$. So our FOC becomes:

$$u'(w + (1 - q)\alpha^* - D) = u'(w - \alpha^* q).$$

- Since the agent is strictly risk adverse, the marginal utility of wealth, $u'(x)$, is strictly decreasing as x gets larger. Thus the slopes can only be equal at one point. Thus,

$$w + (1 - q)\alpha^* - D = w - \alpha^* q \Longrightarrow \alpha^* = D.$$

- So if the insurance premium is actuarily fair ($q = \pi$), a strictly risk averse agent will choose to have FULL insurance.

- Any time we have a risk adverse agent and a risk neutral agent (the insurance company simply maximizes profits), there are gains from trade. In fact, it can be shown that any level of guaranteed income between the agent's certainty equivalence and their expected utility of wealth (without insurance) can be supported as a pareto optimum. The difference between these two quantities is the risk premium, ρ, defined last lecture. It's the most the agent would be willing to pay to eliminate his risk of falling off a very high building.

16 Lecture 16: April 12, 2005

16.1 Measuring Risk Aversion

Local Risk Aversion

- **Definition** Given a C^2 Bernoulli utility function, $u(\cdot)$, the Arrow-Pratt (AP) Measure of Absolute Risk Aversion at wealth x is:

$$r_A = \frac{-u''(x)}{u'(x)}.$$

  So agent 2 is more risk averse than agent 1 if $r_A^2 > r_A^1$. Note this measure allows us to compare attitudes towards risky situations whose outcomes are absolute gains or losses. The measure is only local so the sign of the relationship might be different at different levels of wealth.

$$\text{Higher } r_A \Longrightarrow \text{More Risk Averse.}$$

- Why don't we just use $u''(x)$ since this alone captures the degree of risk aversion? Both to make it comparable to other measures of risk aversion and (more importantly), the AP measure is invariant to affine transformations to the utility function. Given utility function $u(x)$ with AP measure $-u''/u'$, consider:

$$v(x) = \beta u(x) + \gamma.$$

  Then the AP measure is $-\beta u''/\beta u' = -u''/u'$. If we didn't divide by u', these measures of risk aversion would be different.

Global Risk Aversion

- Given C^2 Bernoulli utility functions for individuals 1 and 2: u_1, u_2. Individual 2 is Globally more risk averse than individual 1 if and only if there exists a concave function, $\Psi(\cdot)$ such that:

$$u_2(x) = \Psi(u_1(x)).$$

  That is, u_2 is a concave transform of u_1, eg $u_2(x) = \sqrt{u_1(x)}$. This just means u_2 is more concave than u_1.

Risk Premium and Certainty Equivalent

- Individual 2 is More Risk Averse than individual 1 if and only if:

$$c(F, u_2) \leq c(F, u_1) \text{ for every lottery } F(\cdot).$$

  So $CE_2 \leq CE_1$. Intuitively, this means that agent 2 is willing to accept less money for sure than agent 1 to avoid any risk. Agent 2 is less comfortable with the risk.

- Equivalently, since the risk premium, $\rho = EV - CE$, agent 2 is more risk averse than agent 1 when agent 2's risk premium is higher:

$$\rho(F, u_2) \geq \rho(F, u_1) \text{ for every lottery } F(\cdot).$$

See G-16.1.

- **Pratt's Theorem** The three previous measures of risk aversion are equivalent. Agent 2 is more risk averse than agent 1 if:

$$\text{Arrow Pratt: } r_A^2(x, u_2) > r_A^1(x, u_1) \; \forall \; x \in [a, b]$$

OR

$$\text{Concave Transform: } u_2(x) = \Psi[u_1(x)]$$

OR

$$\text{Certainty Equivalent: } c(F, u_2) \leq c(F, u_1) \text{ for every lottery } F(\cdot).$$

Note we had to make the A-P measure global to make it comparable to the others. Note that last measure is equivalent to:

$$\rho(F, u_2) \geq \rho(F, u_1) \text{ for every lottery } F(\cdot).$$

- **Definition** The Bernoulli utility function, $u(\cdot)$, exhibits decreasing (constant) (increasing) absolute risk aversion if $r_A(x, u)$ is decreasing (constant) (increasing) in x. For example, given $w_2 > w_1$, then an individual's utility function exhibits DARA iff:

$$r_A(w_1 + x, u) > r_A(w_2 + x, u).$$

So DARA if as wealth rises, the person becomes less risk averse ($r_A \downarrow$). With exponential utility, $u(x) = -e^{-\lambda x}$, $\lambda > 0$, $-u''/u' = \lambda$, constant. So an individual with exponential utility has a utility function that exhibits CARA.

- **Relative Risk Aversion** Given C^2, $u(\cdot)$, the coefficient of relative risk aversion at wealth x is defined as:

$$r_R = -x\frac{u''(x)}{u'(x)} = xr_A.$$

Thus,

$$r_R = x\frac{-u''(x)}{u'(x)} = \frac{du'(x)}{dx} \cdot \frac{-x}{u'(x)} = \frac{du'(x)/u'(x)}{-dx/x} = -\frac{\%\Delta u'}{\%\Delta x}.$$

So if r_R is decreasing in x, then we say that an individual's utility function exhibits decreasing relative risk aversion. She would become less risk averse with respect to gambles that are proportational to her wealth as her wealth increases. So, compare a 10 percent loss of income when your wealth is $50,000$ dollars to a 10 percent loss when your wealth is $100,000$ dollars.

16.2 Portfolio Choice

- Consider an agent facing two assets, one safe and one risky. The safe asset pays 1$ per dollar invested and the risky asset pays $z\$ \in [a, b]$, where $F(z)$ is the distribution of z, $E[z] > 1$, and $a < 1$.

- The individual has utility function, $u(\cdot)$, initial wealth, w, and invests some dollar amount, α, in the risky asset (leaving $w - \alpha$ to invest safely).

- Thus the indivdual's end of period wealth would be:

$$(w - \alpha) + \alpha z = w + (z - 1)\alpha.$$

- The problem of the agent is thus:

$$Max_\alpha \, V(\alpha; w) = \int_a^b u[w + (z - 1)\alpha] f(z) dz,$$

  subject to:

$$\alpha \in [0, w].$$

- Let α^* be the arg max of this problem. We will show first that $\alpha^* > 0$ and then show how α^* varies with both the level of risk aversion of the agent and his wealth.

- Kuhn-Tucker FOC:

$$V_\alpha = \phi(\alpha^*; w) = \int_a^b (z - 1) u'[w + (z - 1)\alpha^*] f(z) dz \begin{cases} < 0 & \text{if } \alpha^* = 0 \\ = 0 & \text{if } \alpha^* \in (0, w) \\ > 0 & \text{if } \alpha^* = w \end{cases}$$

  See G-16.2. So $\phi = 0$ only when we have an interior solution.

- Check the SOC:

$$\phi'(\alpha^*; w) = \int_a^b (z - 1)^2 u''[w + (z - 1)\alpha^*] f(z) dz > 0,$$

  so we have a maximum.

- **Proposition** If a risk is actuarily favorable ($E[z] > 1$), then any risk averter will always accept at least a small amount of the risky asset (ie, $\alpha^* > 0$).
  Proof: Consider the FOC evaluated at $\alpha^* = 0$:

$$\phi(0; w) = \int_a^b (z-1) u'[w + (z-1) * 0] f(z) dz = \int_a^b (z-1) u'[w] f(z) dz = u'(w) * E[z-1] > 0.$$

  This contradicts our KT condition which says $\phi < 0$ when $\alpha^* = 0$. So $\alpha^* > 0$.

- Now want to see how α^* varies with the level of risk aversion. Consider individual 2 strictly more risk averse than individual 1. We will show $\alpha_1^* > \alpha_2^*$. Suppose both individuals have initial wealth, w. Also assume $\alpha_1^*, \alpha_2^* < w$. The FOC's for these two individuals are:

$$\phi_1(\alpha_1^*; w) = \int_a^b (z-1)u_1'[w + (z-1)\alpha_1^*]f(z)dz = 0.$$

$$\phi_2(\alpha_2^*; w) = \int_a^b (z-1)u_2'[w + (z-1)\alpha_2^*]f(z)dz = 0.$$

Recall the SOC was negative which means ϕ_i, $i = 1, 2$ are strictly decreasing functions. To prove that $\alpha_1^* > \alpha_2^*$, it is the same as showing that:

$$\phi_2(\alpha_1^*; w) < 0.$$

See G-16.3. Note that $\phi_2(\alpha_2^*; w) = 0$, so to the right of α_2^*, ϕ_2 must be negative. If α_1^* is to the right of α_2^*, then $\phi_2(\alpha_1^*; 2) < 0$.

- More next time.

17 Lecture 17: April 14, 2005

17.1 More on the Demand for Risky Assets

- Recall, last time we were trying to show $\alpha_1^* > \alpha_2^*$ which is the same as showing $\phi_2(\alpha_1^*; 2) < 0$.

- Since agent 2 is strictly more risk averse than agent 1, there is a strictly increasing and concave function, Ψ, such that $u_2(x) = \Psi(u_1(x))$. Thus agent 2's objective function is:

$$V_2(\alpha; w) = \int_a^b \Psi(u_1(w + (z-1)\alpha))f(z)dz.$$

FOC:

$$V_2' = \phi_2(\alpha; w) = \int_a^b (z-1)\Psi'(u_1(w + (z-1)\alpha))u_1'[w + (z-1)\alpha]f(z)dz.$$

Evaluated at α_1^*:

$$V_2' = \phi_2(\alpha_1^*; w) = \int_a^b (z-1)\Psi'(u_1(w + (z-1)\alpha_1^*))u_1'[w + (z-1)\alpha_1^*]f(z)dz.$$

- Note that Ψ' is a positive and decreasing function of z. So consider splitting the integral into two parts:

$$\int_a^1 (z-1)\Psi'(u_1(w + (z-1)\alpha_1^*))u_1'[w + (z-1)\alpha_1^*]f(z)dz+$$

$$+\int_1^b (z-1)\Psi'(u_1(w + (z-1)\alpha_1^*))u_1'[w + (z-1)\alpha_1^*]f(z)dz.$$

We know also that

$$\phi_1(\alpha_1^*; w) = \int_a^b (z-1)u_1'[w + (z-1)\alpha_1^*]f(z)dz = 0.$$

So the Ψ' term is like a weighting term in the first expression. For $z \in [a, 1]$, $(z-1)$ is negative and since Ψ' is a decreasing function of z, for smaller z's, ie, less than 1, the weight is relatively large. For $z \in [1, b]$, $(z-1)$ is positive but the weight is relatively smaller because Ψ' is smaller. Since, overall, the entire integral without the funny weightings equals 0, (ie, the negative $z-1$'s exactly cancel the positive $z-1$'s) we must be placing more weight on negative terms so:

$$\phi_2(\alpha_1^*; w) < 0 \Longrightarrow \alpha_1^* > \alpha_2^*.$$

- Consider a second question. How does investment in the risky asset vary with the agent's level of wealth.

- **Proposition** If the utility function exhibits DARA (CARA) (IARA), then $d\alpha^*/dw > 0$ $(= 0)$ (< 0).
Proof: Recall:

$$sign\ \frac{d\alpha^*}{dw} = sign\ \frac{\partial^2 V(\alpha; w)}{\partial\alpha\partial w}\bigg|_{\alpha=\alpha^*}.$$

Thus, to sign the change in α^* from a change in wealth, we consider the cross partial of the value function. So we have:

$$\begin{aligned}\frac{\partial^2 V(\alpha^*; w)}{\partial\alpha\partial w} = \frac{\partial\phi(\alpha^*; w)}{\partial w} &= \int_a^b (z-1)u''(w+(z-1)\alpha^*)f(z)dz \\ &= \int_a^b (z-1)u''(w+(z-1)\alpha^*)[\frac{u'(w+(z-1)\alpha^*)}{u'(w+(z-1)\alpha^*)}]f(z)dz \\ &= -\int_a^b (z-1)u'(w+(z-1)\alpha^*)r_A(w+(z-1)\alpha^*)f(z)dz\end{aligned}$$

- Now consider the case of where $u(\cdot)$ exhibits DARA. So $\frac{\partial r_A(x)}{\partial x} < 0$. Recall the FOC for the agent:

$$\phi(\alpha^*; w) = \int_a^b (z-1)u'(w+(z-1)\alpha^*)f(z)dz = 0.$$

So we have the same analysis as before. Let's split the integral:

$$-[\int_a^1 (z-1)u'(w+(z-1)\alpha^*)r_A(w+(z-1)\alpha^*)f(z)dz +$$

$$+\int_1^b (z-1)u'(w+(z-1)\alpha^*)r_A(w+(z-1)\alpha^*)f(z)dz].$$

So the first $(z-1)$ is negative and r_A is placing relatively more weight on that term. In the second integral, $(z-1)$ is positive but r_A puts relatively less weight on this. So overall, since the FOC tells us the total integral without the weights is zero, the term inside the brackets must be negative. Thus, since we negate:

$$\frac{\partial^2 V(\alpha^*; w)}{\partial\alpha\partial w} > 0 \Longrightarrow \frac{d\alpha^*}{dw} > 0.$$

- Similar arguments show that:

$$\text{DARA: } \frac{d\alpha^*}{dw} > 0.$$

$$\text{CARA: } \frac{d\alpha^*}{dw} = 0.$$

$$\text{IARA: } \frac{d\alpha^*}{dw} < 0.$$

- So, in sum,
  - (1) A risk averse agent will fully insure when insurance is actuarily fair: $q = \pi$.
  - (2) A risk averse agent will always buy at least some of a risky asset if it is actuarily favorable: $E[z] > 1$.
  - (3) An agent that is strictly more risk averse than another will invest less in the risky asset.
  - (4) An agent whose utility exhibits decreasing (constant) (increasing) absolute risk aversion will invest more (the same amount) (less) in a risky asset as her wealth grows.

17.2 Measuring Risk

First Order Stochastic Dominance

- **Definition** Let $G(\cdot)$ and $F(\cdot)$ be two continuous distribution functions on $[0, \infty)$ such that $F(0) = G(0) = 0$. Then F First Order Stocastically Dominates (FOSD) G iff:

$$F(x) \leq G(x) \; \forall \; x.$$

See G-17.1.

- **Proposition** For every nondecreasing function $u : \Re \mapsto \Re$, F FOSD G iff:

$$U(F) = \int u(x)f(x)dx \geq \int u(x)g(x)dx = U(G).$$

If you let $u(x) = x$, this implies that for random variables y and z which follow F and G respectively, $\overline{y} \geq \overline{z}$. Or in non-standard notation,

$$Mean(F) \geq \; Mean(G).$$

- Proof. Recall, integration by parts:

$$\int udv = uv - \int vdu.$$

So let $u = u(x)$ and $v = F(x)$. So,

$$\int u(x)f(x)dx = \int u(x)dF(x) = [u(x)F(x)]_0^\infty - \int u'(x)F(x)dx.$$

$$\int u(x)g(x)dx = \int u(x)dG(x) = [u(x)G(x)]_0^\infty - \int u'(x)G(x)dx.$$

Thus,

$$U(F) - U(G) = \int u'(x)(G(x) - F(x))dx.$$

Since $G(x) \geq F(x)$, $U(F) \geq U(G)$ as long as $u(x)$ is an increasing function.

- **Definition** F Second Order Stocastically Dominates (SOSD) G iff:

  $$\int_0^{\overline{x}} F(s)ds \leq \int_0^{\overline{x}} G(s)ds \; \forall \, \overline{x} \in [0, \infty).$$

  See G-17.2.

- Note that FOSD $\Longrightarrow$ SOSD but not the other way around. See G-17.3 for two distribution functions which provide a case where F SOSD G but F does not FOSD G. In SOSD, F and G can cross.

- **Proposition** If the distribution F SOSD G, then for any non-decreasing, concave, $u(\cdot)$:

  $$\int_0^{\infty} u(x)f(x)dx \geq \int_0^{\infty} u(x)g(x)dx.$$

  So any risk averse agent will choose F (say, the lottery F) over G. We couldn't say this with FOSD. The idea is that F places more weight on higher outcomes than G does.

- Note also that if F SOSD G, then the mean of F is greater than the mean of G.

- **Definition** Consider an outcome x that follows the distribution F. Further randomize x so that $y = x + z$ where z has distribution function $H_x(z)$ with mean zero. Let the resulting distribution of y be $G(y)$. Note the mean of G is the same as the mean of F since H has mean 0. We say that G is a mean preserving spread (mps) of F. See G-17.4.

- Note that if G is a mps of F, then F SOSD G. So G is a mps of F iff:

  $$U(F) = \int u(x)f(x)dx \geq \int u(x)g(x)dx = U(G),$$

  for any concave function $u(x)$.

- So what about just using the variance of the distribution as our measure of risk? Well it turns out that given two distributions with the same mean and variance, an agent with log utility will prefer one to the other, when he should be indifferent.

18 Lecture 18: April 19, 2005

18.1 Measuring Risk

- **Proposition** G is a mean preserving spread of F ONLY IF the variance of F is smaller than the variance of G. See notes.

- Note that with quadratic utility, the mean and variance are all we need:

$$U(F) \geq U(G) \ IFF \ Var_F(x) \leq Var_G(x).$$

- With exponential utility and normally distributed variables, all we need is the mean and variance. Suppose $u(x) = -e^{-\lambda x}$ and $x \sim F \equiv N(\mu, \sigma^2)$. Then:

$$E[U(F)] = -e^{-\lambda[\mu - \frac{1}{2}\lambda^2\sigma^2]}.$$

  I think this is what she means.

- One other problem with expected utility models is the issue of loss aversion. People dislike a ten dollar loss much more than they like a ten dollar gain. Losses resonate more than gains. See G-18.1. The idea is that there is some reference level of wealth and the utility function changes its concavity property on either side.

- **Definition** Diminishing Sensitivity. The marginal change in perceived well-being is greater for changes that are close to one's reference level than for changes that are far away. So the utility function flattens out when you get further away from your reference (possibly current) level of wealth.

18.2 Topic 3: Moral Hazard and Principal-Agent Problems

- The main issue in principal (P) / agent (A) problems is that actions of agents are hidden from other parties to the transaction. In our setup, the P wants the A to perform an action that is costly to the A, but the action itself is not directly observed by the P.

- Examples. shareholders (P) versus firm's manager (A); landowner (P) versus share-cropper (A); manufacturer (P) versus retailer (A).

- We will assume there is a risk-neutral firm owner (P) that only cares about profits. Let $e \in E$ be the level of effort put forth by the A. Profits are a random variable, π, with (continuous) distribution function, $F(\pi|e)$, on the range, $[\underline{\pi}, \overline{\pi}]$, with associated density, $f(\pi|e)$.

- Assumptions:

  - (1) $f(\pi|e) > 0 \ \forall \ \pi \in [\underline{\pi}, \overline{\pi}], \ e \in E$. So no matter what level of effort is put forth, EVERY level of profits is always possible (though the probabilities of each level will change). This means that the P can't observe π and see exactly what effort the A put forth. There will always be the possibility that the A was a good worker but didn't sell any cars because it rained or the A sucked but a bunch of people still bought cars.

  - (2) For $e_1 > e_2$, $F(\pi|e_1) \leq F(\pi|e_2)\ \forall\ \pi \in [\underline{\pi}, \overline{\pi}]$. See G-18.2. This is the definition of e_1 FOSD e_2. This also implies:

$$\int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi|e_1) d\pi \geq \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi|e_2) d\pi,$$

    Or,

$$E[\pi|e_1] \geq E[\pi|e_2].$$

    This means that if effort is costless, the risk neutral P will want the A to put forth high effort since expected profits will be higher.
  - (3) The A has Bernoulli utility function:

$$u(w, e) = \underbrace{v(w)}_{utility\ from\ wage} - \underbrace{g(e)}_{disutility\ from\ effort}, \quad v' > 0, v'' \leq 0, g' > 0.$$

    So the agent could be risk neutral or risk averse depending on the sign of v''.
  - (4) The A has reservation utility $\overline{u}$.

- Informational Assumptions.
  - (1) e is NOT observable to P.
  - (2) π is observable and verifiable. Thus, parties can only contract on π. We denote compensation schedules $w(\pi)$.
  - (3) $F(\pi|e)$ is common knowledge.
  - (4) $u(w, e)$ and $\overline{u}$ are common knowledge.

- So what is the order of events? 1) the P offers the A $w(\pi)$ and 2) the A either accepts or rejects. If he rejects, he gets $\overline{u}$ and we all go home. 3) If A accepts, he chooses a level of effort, e, which generates a distribution function over profits, $F(\pi|e)$. Then 4) the profits are realized and the A is paid according to $w(\pi)$.

- We always solve these problems backwards.

- Step 4: Consider the expected payoff to the A from accepting $w(\pi)$:

$$\int_{\underline{\pi}}^{\overline{\pi}} [v(w(\pi)) f(\pi|e) d\pi] - g(e).$$

- Step 3: If the A accepts, she chooses:

$$\hat{e}(w(\pi)) = arg\ max_e\ \{\int_{\underline{\pi}}^{\overline{\pi}} [v(w(\pi)) f(\pi|e) d\pi] - g(e)\}.$$

- Step 2: Thus, A accepts if:

$$\int_{\underline{\pi}}^{\overline{\pi}} [v(w(\pi))f(\pi|\hat{e})d\pi] - g(\hat{e}) \geq \overline{u}.$$

- Step 1: The P maximizes profits by choosing a wage contract $w(\pi)$ to maximize:

$$E[\pi] = \int_{\underline{\pi}}^{\overline{\pi}} [\pi - w(\pi)]f(\pi|\hat{e})d\pi.$$

- However, in step 1, the P must take into account that if she sets the wage too low, the A will not accept the contract. The P also wants the A to choose a sufficiently high level of effort. These constraints are summarized in the Individual Rationality (IR) and Incentive Compatibility (IC) constraints.

- Consider a simplification where there only two levels of effort possible, $E = \{e_H, e_L\}$. Assume $F(\pi|e_H) \leq F(\pi, e_L)$ and $g(e_H) > g(e_L)$. Thus the Principal's problem is:

$$Max_{w(\pi),\, e \in E} \left\{ \int_{\underline{\pi}}^{\overline{\pi}} [\pi - w(\pi)]f(\pi|e)d\pi \right\},$$

  subject to:

$$\text{IR:} \quad \int_{\underline{\pi}}^{\overline{\pi}} [v(w(\pi))f(\pi|e)d\pi] - g(e) \geq \overline{u},$$

$$\text{IC:} \;\; e = arg\ max_{\tilde{e}} \{ \int_{\underline{\pi}}^{\overline{\pi}} [v(w(\pi))f(\pi|\tilde{e})d\pi] - g(\tilde{e}) \}.$$

- So our two-stage solution method is to find $w(\pi)$ which maximizes the principal's expected profits subject to IR and IC for EACH level of effort. Then we compare profits at each level of effort and pick the contract that yields the highest expected payoff for the principal.

19 Lecture 19: April 21, 2005

19.1 Principal Agent - Full Information Case

- Suppose effort is completely observable (and verfiable) by the P. Then the P's problem is:

$$Max_{w(\pi),e\in E}\int_{\underline{\pi}}^{\overline{\pi}}[\pi - w(\pi|e)]f(\pi|e)d\pi,$$

subject to:

$$\text{IR: }\int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi))f(\pi|e)d\pi - g(e)\geq \overline{u}.$$

Note that we don't need an IC constraint because the P can impose an arbitrarily large penalty for not choosing the effort level that the P wants the A to choose.

- So the P solves the problem at each effort level (e_H, e_L). For a general effort level, e, the problem is equivalent to:

$$Min_{w(\pi)}\int_{\underline{\pi}}^{\overline{\pi}} w(\pi|e)f(\pi|e)d\pi,$$

subject to IR, because mean profits are independent of the wage, since effort is now known.

- **Proposition** Under full information, with a STRICTLY RISK AVERSE AGENT, the optimal wage for each effort level, e, is unique and consists of a fixed wage.
Proof. Consider the lagrangian for this problem:

$$\mathcal{L} = -\int w(\pi|e)f(\pi|e)d\pi + \gamma[\int v(w(\pi))f(\pi|e)d\pi - g(e) - \overline{u}].$$

Thus the FOC with respect to $w(\cdot)$ for a given level of profits (might not be rigerous):

$$-f(\pi|e) + \gamma v'(w^*(\pi|e))f(\pi|e) = 0.$$

Or,

$$\gamma = \frac{1}{v'(w^*(\pi|e))}.$$

Thus, since $v'' < 0$ for a strictly risk averse agent, v' is single valued, which means w^* is unique!

- Note that for a risk loving agent, the SOC would be violated because a constant wage is NOT optimal for him.

- So we have a fixed wage contract,

$$w^*(\pi|e) = w_e^*,$$

where w_e^* solves:

$$v(w_e^*) - g(e) = \overline{u}.$$

Or,

$$w_e^* = v^{-1}(\overline{u} + g(e)).$$

Note that $g(e_H) > g(e_L)$, so by G-19.1, clearly,

$$w_{e_L}^* < w_{e_H}^*.$$

- The next step is for the P to select the optimal effort level for the agent. Since effort is observable, the P can design his wage contract based solely on the effort so P implicitly chooses the level of effort for the A. So,

$$e^* = arg\ max_{e \in E} \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi|e) d\pi - w_e^*.$$

- And finally the contract offered by the P to the A is:

$$w_e = \begin{cases} w_e^* & \text{if } e = e^* \\ -\infty & \text{if } e \neq e^* \end{cases}$$

So the P guarantees that the worker will choose $e = e^*$ and pays him w_e^*.

- So in the end, the risk neutral P has taken over all the risk from the risk averse A. This is the first best solution since the P can do no better than by "insuring" the agent against risk by offering him a flat fixed wage and taking on the risk himself which he is NOT averse to. The agent gets the equivalent of his reservation utility, $\overline{u}$, and the principal makes profit:

$$E[\pi] = \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi|e^*) d\pi - w_e^*.$$

19.2 Principal Agent - Hidden Information Case

Risk Neutral Agent

- Now we assume that effort is unobservable but the agent is risk neutral, ie:

$$v(w) = w.$$

- We will show that the P must make the A face SOME risk in order to give her the incentive to exert high effort.

- **Proposition** When the agent is RISK NEUTRAL, there exists a contract that generates the same effort choice and same expected payoffs for the P and A as when the effort is observable. This contract must therefore be optimal.
Proof: First, suppose such a contract, $w(\pi)$, exists. Since the P can never do any

better than the full information case, this must still be optimal for the P. Consider a wage contract:

$$w(\pi) = \pi - \alpha,$$

where π are the profits of the firm and α is some positive number less than π. The A becomes the "residual claimant" and will choose effort to maximize expect profits net of effort costs. So the Agent's problem:

$$Max\ E[U] = Max_{e \in E} \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi|e) d\pi - \alpha - g(e).$$

So in a way, the P has sold part of the firm to the A (A is a franchise owner). Suppose e^* maximizes this expression for the agent. The principal then chooses α to extract all rents from the agent. Ie, set alpha such that:

$$\alpha = \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi|e^*) d\pi - g(e^*) - \overline{u}.$$

- Thus, once again, the agent receives the equivalent of $\overline{u}$, her reservation utility, and the principal receives the rest of the surplus:

  $$E[\pi] = \alpha = \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi|e^*) d\pi - g(e^*) - \overline{u}.$$

  Or,

  $$E[\pi] = \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi|e^*) d\pi - w_e^*,$$

  which is precisely the same as in the full information case!

- Next we move to a risk averse agent.

20 Lecture 20: April 26, 2005

20.1 Principal Agent - Hidden Information Case

Risk Averse Agent

- Recall with a risk neutral agent, we got the same result as in the full information case. The problem comes from having a risk averse agent.
- Consider a strictly risk averse agent with $v'' < 0$. The principal will evaluate the optimal contract, $w^0(\pi|e)$ for $e = e_L$ and $e = e_H$ and then pick the one that yields the highest expected profits.
- Consider the low effort contract. **Proposition** The principal will use a fixed wage contract to induce e_L such that:

$$w^0(\pi|e_L) = w^*(\pi|e_L) = v^{-1}[\overline{u} + g(e_L)] = w^*_{e_L}.$$

  Proof: If the wage is fixed (does not depend on profits), the agent will, of course, pick low effort since it's less costly. Any non-fixed wage (one that depends on uncertain profits) would have to be higher than the fixed wage because the agent is risk averse. Thus the principal will simply compensate the agent for her reservation utility plus the disutility of effort.

- Consider the high effort contract. The principal's problem is:

$$Min_{w(\pi|e_H)} \underbrace{\int_{\underline{\pi}}^{\overline{\pi}} w(\pi|e_H) f(\pi|e_H) d\pi}_{expected\ wage\ bill},$$

  subject to:

$$\text{IR:} \quad \int_{\underline{\pi}}^{\overline{\pi}} [v(w(\pi|e_H)) f(\pi|e_H)] d\pi - g(e_H) \geq \overline{u},$$

$$\text{IC:} \quad \int_{\underline{\pi}}^{\overline{\pi}} [v(w(\pi|e_H)) f(\pi|e_H)] d\pi - g(e_H) \geq \int_{\underline{\pi}}^{\overline{\pi}} [v(w(\pi|e_H)) f(\pi|e_L)] d\pi - g(e_L).$$

  Thus, our lagrangian:

$$\mathcal{L} = -\int w(\pi|e_H) f(\pi|e_H) d\pi + \gamma[\int v(w(\pi|e_H)) f(\pi|e_H) d\pi - g(e_H) - \overline{u}] +$$

$$+\mu[\int v(w(\pi|e_H)) f(\pi|e_H) d\pi - g(e_H) - \int v(w(\pi|e_H)) f(\pi|e_L) d\pi - g(e_L)].$$

  FOC:

$$-f(\pi|e_H) + \gamma v'(w^0) f(\pi|e_H) + \mu[v'(w^0) f(\pi|e_H) - v'(w^0) f(\pi|e_L)] = 0.$$

$$-1 + \gamma v'(w^0) + \mu[v'(w^0)(1 - \frac{f(\pi|e_L)}{f(\pi|e_H})] = 0.$$

$$\frac{1}{v'(w^0(\pi))} = \gamma + \mu[1 - \frac{f(\pi|e_L)}{f(\pi|e_H}].$$

- **Proposition** $\gamma > 0$ and $\mu > 0$, ie both IR and IC constraints bind at the optimum. Proof.
  - First, Suppose $\gamma = 0$. Since we have assumed $F(\pi|e_H)$ FOSD $F(\pi|e_L)$, ie the CDF for high effort is under the CDF for low effort, there must be some level of profits such that $f(\pi|e_L) > f(\pi|e_H)$. Note that $f(\pi|e)$ is the slope of $F(\pi|e)$. Thus, as shown in G-20.1, clearly the slope of the the CDF on top must be greater than the CDF on bottom for some range of profits. But this implies:

    $$1 - \frac{f(\pi|e_L)}{f(\pi|e_H)} < 0.$$

    Since $\mu \geq 0$, this means $v'(w^0(\pi)) \leq 0$, which contradicts $v' > 0$. Hence $\gamma > 0$.
  - Now suppose $\mu = 0$. The FOC simplifies to:

    $$\frac{1}{v'(w^0(\pi))} = \gamma.$$

    Since $v'' < 0$, v' is single valued which means $w^0(\pi) = w^*$, fixed!! But this would imply that the agent exerts low effort as was reasoned above. Thus we have a contradiction. Thus $\mu > 0$.
- Note that since the IR constraint binds, the agent will earn expected utility of: $\overline{u} + g(e_H)$, which is the same as before. The loss will be on the principal.
- We can't say much about the high effort wage contract but we do have a few properties.
- **Proposition** Consider $\pi_2 > \pi_1$. Then $w(\pi_2) > w(\pi_1)$ iff:

  $$\frac{f(\pi_1|e_L)}{f(\pi_1|e_H)} > \frac{f(\pi_2|e_L)}{f(\pi_2|e_H)}.$$

  First of all, these are likelihood ratios. This says that if we compare a high and low profit level, the weight placed on observing the high level of profits under low effort must fall and the weight placed on the high level of profits under high effort must rise. So it's relatively more likely for the agent to have worked hard if we realized π_2 then if we saw π_1. This result is called the Monotone Likelihood Ratio Property (MLRP). Proof: Consider $\pi_2 > \pi_1$ with $w(\pi_2) > w(\pi_1)$. Then:

  $$\left(\frac{1}{v'(w^0(\pi_2))} - \frac{1}{v'(w^0(\pi_1))}\right) = \mu\left(\frac{f(\pi_1|e_L)}{f(\pi_1|e_H)} - \frac{f(\pi_2|e_L)}{f(\pi_2|e_H)}\right).$$

So,

$$\pi_2 > \pi_1 \Longrightarrow v'(\pi_2) < v'(\pi_1) \Longrightarrow 1/v'(\pi_2) > 1/v'(\pi_1),$$

so the LHS is positive. Since $\mu > 0$, this means the term in parens is positive which proves the MLRP. Note we get both directions of the proof from this equality.

- **Remark** FOSD does not imply MLRP. You could have one CDF above the other but the two have the same slope at some point which would invalidate the MLRP.

- **Proposition** The principal pays higher expected wages to a strictly risk averse agent when effort is unobservable:

$$\underbrace{\int_{\underline{\pi}}^{\overline{\pi}} w^0(\pi|e_H)f(\pi|e_H)d\pi}_{\textit{expected wage bill}} > w^*_{e_H}.$$

Proof: By Jensen's inequality:

$$v[\int w^0(\pi|e_H)f(\pi|e_H)d\pi] > \int v(w^0(\pi|e_H))f(\pi|e_H)d\pi.$$

But the RHS is just the expected utility of the agent (because IR binds) which equals $\overline{u} + g(e_H) = v(w^*_{e_H})$. Thus,

$$v[\int w^0(\pi|e_H)f(\pi|e_H)d\pi] > v(w^*_{e_H}).$$

Since $v' > 0$,

$$\int w^0(\pi|e_H)f(\pi|e_H)d\pi] > w^*_{e_H},$$

which is what we wanted to show.

- **Remark** The difference in the wage bill is the cost of information.

- Solution to the Principal's problem. The principal will choose $w^0(\pi|e_H)$ over $w^*_{e_L}$ iff:

$$\int_{\underline{\pi}}^{\overline{\pi}} (\pi - w^0(\pi|e_H))f(\pi|e_H)d\pi > \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi|e_H)d\pi - w^*_{e_L}.$$

Ie,

$$E[\pi|w^0(\pi|e_H)] > E[\pi|w^*_{e_L}].$$

21 Lecture 21: April 28, 2005

21.1 Finishing up the Hidden Action Case with a Risk Averse Agent

- Note that the principal will sometimes find it optimal not to offer the high effort contract because it's simply too expensive to make the agent work hard.
- So to summarize when we ONLY HAVE TWO LEVELS OF EFFORT:
  - (1) The principal will use a fixed wage contract to implement low effort (ie, set the IR constraint to zero).
  - (2) To implement $w^*(\pi|e_H)$, both the IR and IC constraints will bind so you can pin down the wage contract from these two constraints alone.
  - (3) For a high effort contract where $\pi_1 > \pi_2$,

$$w(\pi_1) > w(\pi_2) \text{ iff } \frac{f(\pi|e_L)}{f(\pi|e_H)} \text{ is decreasing in } \pi \ .$$

  - (4) The principal pays higher expected wages to a strictly risk averse agent when effort is UNobservable than when effort is observable.
- Note for finite levels of effort larger than 2, we don't get such nice results. Some of these IC constraints may NOT bind.

21.2 Continuous Effort Case

- Suppose now that $e \in E = [\underline{e}, \overline{e}]$. The optimal contract, $w^0(\pi|e)$, for implementing effort level e solves the following:

$$Min_{w(\pi|e)} \int_{\underline{\pi}}^{\overline{\pi}} w(\pi|e) f(\pi|e) d\pi,$$

subject to:

$$IR: \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e)) f(\pi|e) d\pi - g(e) \geq \overline{u},$$

$$IC: \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e)) f(\pi|e) d\pi - g(e) \geq \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e)) f(\pi|\tilde{e}) d\pi - g(\tilde{e}) \ \forall \ \tilde{e} \in \{E/e\}.$$

So for this problem, there are an infinite number of IC constraints! So we might get a solution, many solutions, or there might even not be any wage contract which satisfies all the constraints simultaneously.

- Here's one way we might solve this problem. Consider the agent facing a wage contract offered by the principal: $w(\pi|e)$. His problem becomes:

$$Max_{e \in E} \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f(\pi|e)d\pi - g(e).$$

Noting that the agent does not control the way his choice of effort impacts his wage in this problem, the FOC is:

$$\int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f_e(\pi|e)d\pi - g'(e) = 0.$$

Note that the First-Order IC condition (FOIC) will only be satisisfied when the agent's choice of e corresponds to the effort level the wage contract is based upon.

- Given this, we can replace the infinity of IC constraints with the FOIC because we know it must hold for any wage contract the principal offers. Thus the problem of the principal becomes:

$$Min_{w(\pi|e)} \int_{\underline{\pi}}^{\overline{\pi}} w(\pi|e)f(\pi|e)d\pi,$$

subject to:

$$IR: \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f(\pi|e)d\pi - g(e) \geq \overline{u},$$

$$FOIC: \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f_e(\pi|e)d\pi - g'(e) = 0.$$

Thus, our lagrangian is:

$$\mathcal{L} = -\int_{\underline{\pi}}^{\overline{\pi}} w(\pi|e)f(\pi|e)d\pi +$$

$$+\gamma[\int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f(\pi|e)d\pi - g(e) - \overline{u}] +$$

$$+\mu[\int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f_e(\pi|e)d\pi - g'(e)].$$

FOC:

$$-f(\pi|e) + \gamma v'(w(\pi|e))f(\pi|e) + \mu v'(w(\pi|e))f_e(\pi|e) = 0.$$

Or,

$$\frac{1}{v'(\hat{w}(\pi|e))} = \gamma + \mu \left[\frac{f_e(\pi|e)}{f(\pi|e)}\right].$$

- As before, we can show that $\gamma, \mu > 0$. Also, the ratio of f_e to f is the same term we had before, but now in continuous form. Thus, the Monotonic Likelihood Ratio Property is now in terms of this ratio.

- To summarize, in the continuous case, the choice of e that satisfies the FOIC condition need not be unique, or even a maximium. To insure that the effort level that satisfies the FOIC condition is indeed a global max we need:

$$\int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f(\pi|e)d\pi - g(e)$$

to be globally concave in e. Pretty significant restriction.

- A sufficient condition for this is (1) $F(\pi|e)$ satisfies the MLRP and (2) $F(\pi|e)$ is convex in e.

21.3 Topic 4: Adverse Selection, Signaling, and Screening

- We now are operating in a situation where one agent has some hidden ability or characteristic. Examples might be (1) a worker knows his productivity and the employer doesn't, (2) an individual knows he's got 2 days to live and the life insurer doesn't, (3) a car dealer knows the quality of a car but the buyer doesn't (Akerlof's Lemons), (4) the buyer knows his valuation of an item for sale and the seller doesn't.

- With hidden information, the market outcome will generally be inefficient and this is called the "adverse selection problem".

- Solutions to the problem include Signaling and Screening.

The Lemon's Problem: A Formal Treatment

- We have many firms with CRTS techology who are risk neutral and sell a product with a price of 1.

- We have many workers (normalized to 1) who are risk neutral who have productivity/ability level:
$$\theta \sim F(\cdot) \; on \; [\underline{\theta}, \overline{\theta}] \subset \Re.$$

- $f(\theta)$, which due to poor notation is not a density, is the contribution to a firm of a worker with productivity θ. $r(\theta)$ is the opportunity cost of a worker of type θ of accepting employment at one of the firms. Kind of a "self-employment" return.

- Assume $f(\theta) = \theta$. So my type is such that if I work for one hour at a firm, I produce θ units of the good.

- We assume perfectly competitive firms who will hire labor until MRP = MC or:

$$\theta = w.$$

The equilibrium wage of a worker with productivity, θ, is:

$$w^*(\theta) = \theta.$$

No reason to pay higher than θ, and if they pay lower than θ, another firm will compete that worker away.

- Thus, only workers who are willing to work for wage, $w^*(\theta) = \theta$, are those for whom:

$$r(\theta) \leq w^*(\theta) = \theta.$$

- Profits to the firm are thus $\theta - w = 0$. Since there is only one worker with productivity θ who produces θ units (sold at \$1 each) and gets paid $w = \theta$.

- Thus, only workers with productivity, $\theta \in [r(\theta), \overline{\theta}]$ are employed.

22 Lecture 22: May 3, 2005

22.1 More on Adverse Selection with Perfect Information

- Recall in the case where θ is observable, the firm charges a wage of $w^*(\theta) = \theta$, makes no profits, and only workers of type $\theta \in [r(\theta), \overline{\theta}]$ are employed. This is Pareto optimal since all workers who are better off working (their productivity is higher than their reservation value), are working and those that are better off in their garage, are not employed.
- See G-22.1. Note that we have not said anything about the shape of $r(\theta)$. If $r(\theta) < \theta$ $\forall\ \theta$, then everyone is employed!

22.2 More on Adverse Selection with Imperfect Information

- Suppose now that θ is private information for the worker. The equilibrium wage must be constant since it cannot depend on θ. Instead the wage will depend on the distribution of productivity CONDITIONAL on a worker accepting work.
- The set of types of workers who accept to work at wage w is:

$$\Theta(w) = \{\theta | r(\theta) \leq w\}.$$

  And the average productivity of those workers is:

$$E[\theta | \theta \in \Theta(w)] = E[\theta | r(\theta) \leq w].$$

- So the equilibrium wage satisfies:

$$w^* = E[\theta | r(\theta) \leq w^*].$$

  So we seek a fixed point. See G-22.2 where we assume $r'(\theta) > 0$. We adjust w^* until the expected productivity of those workers who accept the job exactly equals this wage.

- **Remark** Note that there may be one, many, or NO fixed points to this problem!
- Typically, this equilibrium will NOT be pareto efficient. Consider an equilibrium wage, w^*. Given a worker of productivity θ_A (in graph),

$$\theta_A > w^*.$$

  It may then be true (though it doesn't have to be):

$$\theta_A > r(\theta_A) > w^* \Longrightarrow \text{Not pareto optimal.}$$

  So this guy has a productivity higher than his reservation value but ISN'T working.

- **Remark** The fact that those workers who accept the job at wage w are those from the bottom (least productive) of the distribution is exactly the "Adverse Selection" problem.

Special Case: $r(\theta) = r \in (\underline{\theta}, \overline{\theta})$

- To get pareto optimality, we should get that all workers with $\theta \in [r, \overline{\theta}]$ are employed.
- If $w^* \geq r$, EVERY worker will choose to be employed because the wage is higher than everyone's reservation value. If $w^* < r$, no one will work. Thus,

$$E[\theta | r \leq w] = E[\theta].$$

- In equilibrium,

$$w^* = E[\theta],$$

and,

$$\Theta(w^*) = [\underline{\theta}, \overline{\theta}] \text{ for } w^* \geq r.$$

$$\Theta(w^*) = \emptyset \text{ for } w^* < r.$$

- So if there are many low productivity workers, this will drive down the expected productivity below r so no workers will be hired. If there are a lot of high productivity workers, this drives up the expected productivity and EVERYONE works. Both are pareto suboptimal because too few workers are working in the first case and too many are working in the second case.

General Case: $r'(\theta) > 0$

- Assume $r(\theta) < \theta \ \forall \ \theta$ and $r(\overline{\theta}) > E[\theta]$. Then the equilibrium wage is:

$$w^* = E[\theta | r(\theta) \leq w^*].$$

- To find the equilibrium outcome, find a fixed point. Denote:

$$\phi(w) = E[\theta | r(\theta) \leq w].$$

Then for $w = r(\underline{\theta}), \phi(w) = \underline{\theta}$. So if the wage is set at the reservation value of the lowest productivity worker, then the expected conditional productivity is of course just the productivity of that one worker. If $w = r(\overline{\theta})$, $\phi(w) = E[\theta]$ since the conditioning is no longer binding.

- The equilibrium is, as we said, inefficient, because high productivity workers are not employed.
- **Remark** Observe that for $r'(\theta) > 0$, $r(\theta) < \theta$ and $r(\overline{\theta}) > E(\theta)$:

$$E[\theta] > E[\theta | r(\theta) \leq w^*].$$

So the unconditional mean productivity is always higher than when we condition on the workers choosing to work. We are just chopping off the top of the distribution which drives down the expected value. If $r(\theta)$ is increasing and concave, then essentially no workers are hired. We are just saying that the adverse selection problem can be very severe depending on the underlying distribution of productivity, F.

22.3 Signaling: Spence

- The idea of Signaling is to get around the Adverse Selection problem. We allow workers with a high θ to engage in some costly and observable action that workers of low θ would not want to imitate.

- So consider two types of workers with, $\theta_H > \theta_L > 0$, and let $\lambda = Pr\{\theta = \theta_H\} \in (0,1)$. Let e denote the level of education which is attainable at cost, $c(e,\theta)$. Assume the following conditions hold:

  - (1) $c(0,\theta) = 0$. Being a dumbass is free.
  - (2) $c_e(e,\theta) > 0$, $c_{ee}(e,\theta) > 0$. So the cost function is increasing and convex.
  - (3) $c_\theta(e,\theta) < 0$. The high type obtains education with less cost.
  - (4) $c_{e\theta}(e,\theta) < 0$. So the marginal cost of education is decreasing in your productivity. This is called the Spence-Mirrlees Condition, or the "Single Crossing Property". We will discuss this property next time.

23 Lecture 23: May 5, 2005

23.1 More on Signaling

- To understand the single cross property from last time, see G-23.1. Consider separable utility functions of the form $u(w, e, \theta_L) = w - c(e, \theta_L)$ and a similar utility for the high type. Then, as in G-23.1, the indifference curves of an agent of low and high type can only cross once (if they cross at all). This comes directly from the assumption that $c_{e\theta}(e, \theta) < 0$.
- So continuing with the model. Assume that education does not affect a worker's θ. This is a strange assumption if e stands for education, but that's the way Spence modeled it. Thus a worker's utility is:

$$u(w, e, \theta) = w - c(e, \theta),$$

  and assume $r(\theta_H) = r(\theta_L) = 0$ with $\theta_H > \theta_L > 0$.

- If we didn't have the signaling possibility, all workers would work and the equilibrium wage would be $w^* = E[\theta]$. This is pareto optimal.
- Assume we have two firms with CRTS techonology
- Timing of this game of incomplete information is as follows:
  - ($t = 0$): Nature selects a worker's type which remains private to the worker.
  - ($t = 1$): Workers choose level of e.
  - ($t = 2$): Firms observe e and simultaneously make wage offers to the worker. So the firms cannot cooperate.
- So we seek a Perfect Baysian Equilibrium (PBE) which specifies both strategies (e_L, e_H, and $w(e)$) and beliefs of the agents.
  - (1) Workers will have the strategy of choosing e_L or e_H depending on their realization of θ.
  - (2) Firm's beliefs have to be consistent with workers equilibrium strategies. Thus they have to satisfy Bayes rule whenever possible.
  - (3) Let $\mu(e)$ denote the firm's common belief over the workers type conditional on observed e. Thus $\mu(e)$ is the probability a worker is of the high type. So $\mu(e_L) = 0$ and $\mu(e_H) = 1$. What about a realization like $\tilde{e}$ that is not the high or low level of effort. These are called the "off equilibrium path" beliefs and we have the freedom to assign $\mu(\tilde{e})$ whatever value we want.
- Note that if both a high type and a low type choose the same level of education, this is called a "pooling equilibrium". If they choose different level, we have a "separating equilibrium".

- The equilibrium is perfect because firms will optimize at $t = 2$ when they choose the wage knowing that workers have optimized at $t = 1$ when they choose their education level.

- So a set of strategies (e_L, e_H, and $w(e)$) and the belief function, $\mu(e)$, are a PBE if:

  - (1) The worker's strategy is optimal given the firm's strategies.
  - (2) $\mu(e)$ is derived from the worker's strategies using Bayes Rule to update whenever possible.
  - (3) The firm's wage offers consitute a NE in the simultaneous wage-setting game in which $Pr(\theta_H|e) = \mu(e)$.

24 Lecture 24: May 10, 2005

24.1 More on Signaling

- Recall we have firms with CRS technology who do not observe their worker's productivity, θ. Workers signal with their choice of education.
- Wage offered is $w(e)$, choosen simultaneously by the firms.
- We seek a Perfect Baysian Equilibrium that consists of strategies of workers and firms as well as the beliefs of the firms about the type of the worker. Strategies must be optimal, Bayes rule must be used to update wherever possible, and the the firm's wage offers constitute a NE in the simultaneous wage-setting game.
- We have two types of equilibria: pooling ($e_H = e_L$) and separating ($e_L \neq e_H$).
- In stage 2, there is Bertrand competition between firms which guarantees:

$$w^*(e) = \mu(e)\theta_H + (1 - \mu(e))\theta_L, \quad (*)$$

  which is the expected productivity of a worker with education e.
- Thus the worker will ALWAYS accept this wage since $r(\theta) = 0$.
- In stage 1, the worker knows that the wage she will be offered will depend on her choice of e, so her utility:

$$u(w, e) = w - c(e, \theta).$$

Separating Equilibrium (SE)

- We assume there are only pure strategies (no mixing) and the equilibrium is symmetric (all low types do the same and all high types do the same). Let $e^*(\theta)$ denote the worker's equilibrium choice of education. We seek an equilibrium such that:

$$e^*(\theta_H) \neq e^*(\theta_L).$$

- We know in a SE that $\mu(e^*(\theta_H)) = 1$ and $\mu(e^*(\theta_L)) = 0$.
- Also, in an SE:
  - (1) In any SE: $w^*(e^*(\theta_H)) = \theta_H$ and $w^*(e^*(\theta_L)) = \theta_L$. This follow directly from equation (*) if we substitute in for $\mu(e)$.
  - (2) In any SE: $e^*(\theta_L) = 0$. Since (1) holds, the low guy gets utility $\theta_L - c(e, \theta_L)$ which is minimized for $e = 0$. Note there is a subtlety regarding $e = 0$ being off the equilibrium path, but since there are only two levels of effort, we can safely assume that the worst the worker can do is have the firm think he is a low type ... which he is! So it's safe to say he will completely slack off if he's a low type.

- So we have:

$$w^*(e^*(\theta_L)) = \theta_L.$$
$$w^*(e^*(\theta_H)) = \theta_H.$$
$$e^*(\theta_L) = 0.$$

- So what level of effort should the high guy do? Note that the low type earns utility $U_L = \theta_L - c(0, \theta_L) = \theta_L$. Let $\underline{e}$ be the minimum level of e for the high type that is consistent with a SE. So set:

$$U_L = \theta_L - c(0, \theta_L) = \theta_H - c(\underline{e}, \theta_L). \quad (**)$$

This means that the low type is just indifferent between setting $e = 0$ and getting θ_L and setting $e = \underline{e}$, fooling the firm, and getting $\theta_H - c(\underline{e}, \theta_L)$. So the low type would not deviate. So conjecture:

$$e^*(\theta_H) = \underline{e}.$$

- So the low type will not deviate from $e = 0$, but are we sure the high type would would want to do $\underline{e}$? The high type would NOT deviate if:

$$U_H = \theta_H - c(\underline{e}, \theta_H) \geq \theta_L - c(0, \theta_H).$$

Or,

$$\theta_H - \theta_L \geq c(\underline{e}, \theta_H) - c(0, \theta_H). \quad (***)$$

But recall equation (**) rewritten:

$$\theta_H - \theta_L = c(\underline{e}, \theta_L) - c(0, \theta_L).$$

Plug this into equation (***)

$$c(\underline{e}, \theta_L) - c(0, \theta_L) \geq c(\underline{e}, \theta_H) - c(0, \theta_H).$$

$$c(\underline{e}, \theta_L) \geq c(\underline{e}, \theta_H).$$

Which is TRUE by assumption: $c_\theta(e, \theta) < 0$. So the high type will not deviate. This is also true by the Single Crossing Property.

- So we now have:

$$w^*(e^*(\theta_L)) = \theta_L.$$
$$w^*(e^*(\theta_H)) = \theta_H.$$
$$e^*(\theta_L) = 0.$$
$$e^*(\theta_H) = \underline{e}.$$

- What about off equilibrium path beliefs? We can make these whatever we want as long as they are consistent with a PBE, so consider:

$$\mu(e) = \begin{cases} 0 & \text{for } e < \underline{e} = e^*(\theta_H) \\ 1 & \text{for } e \geq \underline{e} = e^*(\theta_H) \end{cases}$$

And this implies our equilibrium wage schedule:

$$w^*(e) = \begin{cases} \theta_L & \text{for } e < \underline{e} = e^*(\theta_H) \\ \theta_H & \text{for } e \geq \underline{e} = e^*(\theta_H) \end{cases}$$

So neither type has an incentive to deviate. No reason to get education above 0 or $\underline{e}$ for the low and high type respectively, and there is no reason for the high type to deviate below $\underline{e}$ because he gets less utility from being pegged as a slacker.

- **Remark** There exists possibly many SE. Consider $\overline{e}$ to be the highest e consistent with a SE. So it is defined by:

  $$U_H = \theta_H - c(\overline{e}, \theta_H) = \theta_L - c(0, \theta_H).$$

  So this is the highest threshold level of education that if you pushed the high type any further, they would choose zero education and just get the low wage. In general there will be a gap such that $e \in [\underline{e}, \overline{e}]$. So we have a continuum of equilibria. However, these equilibia can be pareto ranked and $\underline{e}$ is the best for the high type.

- See G-24.1 for a diagram of the equilibrium. Note $\underline{e}$ must be on both indifferent curves and the low type's indifference curve must be just touching at the two points in the graph. You can see that as we raise $e^*(\theta_H)$, the wage profile shifts to the right and the high type ends up worse off.

- In general, with NO signaling, the wage is constant across types and equal to:

  $$\overline{w}^* = E[\theta] = \lambda\theta_H + (1 - \lambda)\theta_L.$$

  With signaling, the workers earn:

  $$U_H = w^*(e^*(\theta_H)) - c(e^*(\theta_H), \theta_H) = \theta_H - c(e^*(\theta_H), \theta_H).$$

  $$U_L = w^*(e^*(\theta_L)) - c(e^*(\theta_L), \theta_L) = \theta_L.$$

- So for λ sufficiently high, the high type does WORSE than under the no-signaling case. The low type always does WORSE because $\theta_L < E[\theta]$. So we have a rat race where everyone could be doing worse off but no one can afford to deviate. Staying late in the office because everyone else is staying late.

25 Lecture 25: May 12, 2005

25.1 Finalizing Signaling

- A key result of the signaling analysis is the requirement of a PBE regarding off the equilibrium-path beliefs.
- Note we had a lot of freedom in choosing what sort of probability we wanted to place on actions that were off the equilibrium path and this resulted in multiple equilibria.
- One way to restrict the number of equilibria, is to introduce Intuitive Criterion. This means that we eliminate PBE if it is sustained by beliefs that place positive probability on a type for whom a deviation would be equilibrium dominated. That is, the observation of an agent doing an action that if off the equilibrium path wouldn't be reasonable because that agent would have another dominant strategy. In general terms, think of this as SPNE versus NE. This may in fact eliminate ALL equilibria, but it general it is just called an Equilibrium Refinement.

25.2 Screening

- Whereas in signaling, the informed agent took a costly action to reveal his type, here we have the uninformed agent taking steps to distinguish between types.
- Consider two types with $\theta_H > \theta_L > 0$. Let $\lambda = Pr\{\theta = \theta_H\} \in (0, 1)$. Assume $r(\theta_H) = r(\theta_L) = 0$.
- Different jobs have different task levels, $t \geq 0$, which do NOT affect the productivity of the worker (or the resulting profits of the firm), but do affect a workers utility:

$$u(w, t, \theta) = w - c(t, \theta),$$

  where $c(t, \theta)$ is the amount a worker of type θ suffers when assigned to task t.
- Assume the following properties on the cost of tasks function:
  - (1) $c(0, \theta) = 0$.
  - (2) $c_t(t, \theta) > 0$; $c_{tt}(t, \theta) > 0$.
  - (3) $c_\theta(t, \theta) < 0$; $c_{t\theta}(t, \theta) < 0$ which gives us single crossing.
- There are two firms with CRS technology.
- Timing. At t_0, nature selects worker's type which is either public or private information. At t_1, firms simultaneously announce a set of contracts, (w, t). At t_1, workers select their prefered contract.
- In the case of ties, workers pick the contract with the lesser task or just randomize.

- In equilibrium, we seek a Subgame Perfect Equilibrium in pure strategies where (w_H, t_H) and (w_L, t_L) are the contracts designed to be intended for the high and low types respectively.

First Best - Types are Public Information

- In equilibrium,

$$t_H^* = t_L^* = 0,$$
$$w_H^* = \theta_H,$$
$$w_L^* = \theta_L.$$

Why? Because first agents must get paid their productivity as we have argued before. The market is competitive which means that wages can't go below their productivity and there's no need for them to be higher. Since the level of the task ONLY reduced utility, and does not increase productivity, then the level of the task acts as a negative wage. So there is no reason (in the full information case) for the task to be different from zero.

Second Best - Types are Private Information

- We have five necessary conditions that characterize the SPE in this private information game:
  - (1) In any SPE, firms must earn zero profits.
  - (2) Any SPE must be a separating equilibrium with $(w_H, t_H) \neq (w_L, t_L)$.
  - (3) In a separating equilibrium, $w_L^* = \theta_L$ and $w_H^* = \theta_H$.
  - (4) In a separating equilibrium, $t_L^* = 0$.
  - (5) In a separating equilibrium, $t_H^* = \hat{t}$ which satisfies:

    $$w_L^* - t_L^* = \theta_L = \theta_H - c(\hat{t}, \theta_L).$$

    So the low type will be just indifferent between accepting $(w_L^*, 0)$ and accepting $(w_H^*, \hat{t}) = (\theta_H, \hat{t})$.

- Proofs:
  - (1) This is just a bertrand argument that if any firm is making positive profits, the other firm can just raise his wage by a small amount above the others and steal all the profits.
  - (2) Suppose there was a pooling equilibrium at (w^p, t^p). A firm would have an incentive to deviate to a wage and task level just above these pooling values thereby attracting ONLY high type workers (SCP) and earn positive profits.

- (3) This again is bertrand reasoning. We know that $w_L^* \geq \theta_L$ and $w_H^* \geq \theta_H$ but since the firms make zero profits (from (1)), it must be that $w_L^* = \theta_L$ and $w_H^* = \theta_H$.
- (4) For any contract with a positive task, there would be another contract with a lesser task and a slightly lower wage that makes the worker strictly better off and the firm have positive profits.
- (5) This is really the heart of the screen equilibrium. If we set $t_H^* < \hat{t}$, then low productivity workers would find it optimal to accept this (high) contract. So that can't happen. If we set $t_H^* > \hat{t}$, then there would be some contract with a lesser task and slightly smaller wage that would yield the worker strictly higher utility and the firm positive profits. Note this task must still be such that the low type workers would not accept it.

- So, in summary, workers are paid their productivity and $t_L^* = 0$, but:

$$t_H^* > 0.$$

So we have a situation where the task is disorting the otherwise first best equilibrium. High productivity workers are engaging in a costly task only to distinguish themselves from the low types (Loss in welfare).

- Note that a separating equilibrium exists for a low λ, but NOT for a high λ. That is, if there are lot of high productivity types in the economy, the firm will not find it worthwhile to screen to separate. If most workers are slackers, then it will be worthwhile to screen.

26 Final Review

26.1 Key Lectures Notes

- Preference Axioms over lotteries: Completenes, Transitivity, Continuity: $L \succ L' \succ L''$ implies
$$\alpha L + (1-\alpha)L'' \sim L',$$
and Independence:
$$L \succ L' \ iff \alpha(L) + (1-\alpha)L'' \succ \alpha L' + (1-\alpha)L''.$$
- Expected utility form: $U(L) = \sum_{n=1}^{N} u_n p_n$, which is called the von Neumann-Morganstern utility function.
- Expected Utility theorem: Suppose prefs satisfy axioms. Then there exists a utility representation of the preferences in which we can assign numbers to the N outcomes so that $L \succ L'$ iff $\sum_n u_n p_n > \sum_n u_n p'_n$.
- VNM utility functions are equivalent only through affine transforms.
- Certainty Equivalence: $u(c) = \int u(x)f(x)dx = u(F)$.
- Risk premium: $\rho = EV - CE = \int xf(x)dx - c$. Maximum amount an agent would pay to avoid risk.
- Risk aversion: $E[u(x)] \leq u[E(x)]$.
- If an insurance premium is actuarily fair ($q = \pi$), a strictly risk averse agent will choose full insurance.
- Arrow-Pratt Absolute risk aversion, $r_A = -u''(x)/u'(x)$, Relative: $r_R = -xu''(x)/u'(x)$. Higher r_A, more risk averse.
- Global Risk Aversion (Pratt's Theorem): Agent 2 is MORE RISK AVERSE than agent 1 if (any – which implies all):
$$r_A^2 > r_A^1,$$
$$u_2(x) = \Psi[u_1(x)],$$
$$c(F, u_2) \leq c(F, u_1) \Rightarrow \rho(F, u_2) \geq \rho(F, u_1).$$
- If a risk is actuarily favorable ($E[z] > 1$), then any risk averter will always accept at least a small amount of the risky asset.
- A more risk averse agent will invest less in a risky asset than a riskier agent.
- First Order Stochastic Dominance. F FOSD G if $F(x) \leq G(x)\ \forall x$. This implies:
$$U(F) \int u(x)f(x)dx \geq \int u(x)g(x)dx = U(G).$$

- Second Order Sotchastic Dominance. F SOSD G if:

$$\int_0^m F(s)ds \leq \int_0^m G(s)ds \; \forall \; m \in [0, \infty).$$

- FOSD implies SOSD but not vice versa.

Moral Hazard and P/A Problems

- Thus the Principal's problem is:

$$Max_{w(\pi),\, e \in E} \left\{ \int_{\underline{\pi}}^{\overline{\pi}} [\pi - w(\pi)] f(\pi|e) d\pi \right\},$$

subject to:

$$\text{IR: } \int_{\underline{\pi}}^{\overline{\pi}} [v(w(\pi)) f(\pi|e) d\pi] - g(e) \geq \overline{u},$$

$$\text{IC: } e = arg\ max_{\tilde{e}} \{ \int_{\underline{\pi}}^{\overline{\pi}} [v(w(\pi)) f(\pi|\tilde{e}) d\pi] - g(\tilde{e}) \}.$$

- So our two-stage solution method is to find $w(\pi)$ which maximizes the principal's expected profits subject to IR and IC for EACH level of effort. Then we compare profits at each level of effort and pick the contract that yields the highest expected payoff for the principal.

- Full information case: P's problem:

$$Max_{w(\pi), e \in E} \int_{\underline{\pi}}^{\overline{\pi}} [\pi - w(\pi|e)] f(\pi|e) d\pi,$$

subject to:

$$\text{IR: } \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi)) f(\pi|e) d\pi - g(e) \geq \overline{u}.$$

Or,

$$Min_{w(\pi)} \int_{\underline{\pi}}^{\overline{\pi}} w(\pi|e) f(\pi|e) d\pi,$$

subject to IR.

- Under full information, with a STRICTLY RISK AVERSE AGENT, the optimal wage for each effort level, e, is unique and consists of a fixed wage.

- Hidden Information Case. Risk Neutral Agent.

- When the agent is RISK NEUTRAL, there exists a contract that generates the same effort choice and same expected payoffs for the P and A as when the effort is observable. This contract must therefore be optimal.

- Hidden Information Case. Risk Averse Agent.
- Consider the low effort contract. **Proposition** The principal will use a fixed wage contract to induce e_L such that:

$$w^0(\pi|e_L) = w^*(\pi|e_L) = v^{-1}[\overline{u} + g(e_L)] = w^*_{e_L}.$$

- Consider the high effort contract. The principal's problem is:

$$Min_{w(\pi|e_H)} \underbrace{\int_{\underline{\pi}}^{\overline{\pi}} w(\pi|e_H) f(\pi|e_H) d\pi}_{expected\ wage\ bill},$$

subject to:

$$\text{IR:} \quad \int_{\underline{\pi}}^{\overline{\pi}} [v(w(\pi|e_H)) f(\pi|e_H)] d\pi - g(e_H) \geq \overline{u},$$

$$\text{IC:} \quad \int_{\underline{\pi}}^{\overline{\pi}} [v(w(\pi|e_H)) f(\pi|e_H)] d\pi - g(e_H) \geq \int_{\underline{\pi}}^{\overline{\pi}} [v(w(\pi|e_H)) f(\pi|e_L)] d\pi - g(e_L).$$

- Note that since the IR constraint binds, the agent will earn expected utility of: $\overline{u} + g(e_H)$, which is the same as before. The loss will be on the principal.
- **Proposition** Consider $\pi_2 > \pi_1$. Then $w(\pi_2) > w(\pi_1)$ iff:

$$\frac{f(\pi_1|e_L)}{f(\pi_1|e_H)} > \frac{f(\pi_2|e_L)}{f(\pi_2|e_H)}.$$

This result is called the Monotone Likelihood Ratio Property (MLRP).

- **Proposition** The principal pays higher expected wages to a strictly risk averse agent when effort is unobservable:

$$\underbrace{\int_{\underline{\pi}}^{\overline{\pi}} w^0(\pi|e_H) f(\pi|e_H) d\pi}_{expected\ wage\ bill} > w^*_{e_H}.$$

- Solution to the Principal's problem. The principal will choose $w^0(\pi|e_H)$ over $w^*_{e_L}$ iff:

$$\int_{\underline{\pi}}^{\overline{\pi}} (\pi - w^0(\pi|e_H)) f(\pi|e_H) d\pi > \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi|e_H) d\pi - w^*_{e_L}.$$

Ie,

$$E[\pi|w^0(\pi|e_H)] > E[\pi|w^*_{e_L}].$$

- So to summarize when we ONLY HAVE TWO LEVELS OF EFFORT:

- (1) The principal will use a fixed wage contract to implement low effort (ie, set the IR constraint to zero).
- (2) To implement $w^*(\pi|e_H)$, both the IR and IC constraints will bind so you can pin down the wage contract from these two constraints alone.
- (3) For a high effort contract where $\pi_1 > \pi_2$,

$$w(\pi_1) > w(\pi_2) \text{ iff } \frac{f(\pi|e_L)}{f(\pi|e_H)} \text{ is decreasing in } \pi \ .$$

- (4) The principal pays higher expected wages to a strictly risk averse agent when effort is UNobservable than when effort is observable.

- Continuous Effort Case. The optimal contract, $w^0(\pi|e)$, for implementing effort level e solves the following:

$$Min_{w(\pi|e)} \int_{\underline{\pi}}^{\overline{\pi}} w(\pi|e)f(\pi|e)d\pi,$$

subject to:

$$IR: \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f(\pi|e)d\pi - g(e) \geq \overline{u},$$

$$IC: \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f(\pi|e)d\pi - g(e) \geq \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f(\pi|\tilde{e})d\pi - g(\tilde{e}) \; \forall \; \tilde{e} \in \{E/e\}.$$

- Consider the agent facing a wage contract offered by the principal: $w(\pi|e)$. His problem becomes:

$$Max_{e\in E} \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f(\pi|e)d\pi - g(e).$$

FOC is:

$$\int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f_e(\pi|e)d\pi - g'(e) = 0.$$

Note that the First-Order IC condition (FOIC)

- Given this, we can replace the infinity of IC constraints with the FOIC because we know it must hold for any wage contract the principal offers. Thus the problem of the principal becomes:

$$Min_{w(\pi|e)} \int_{\underline{\pi}}^{\overline{\pi}} w(\pi|e)f(\pi|e)d\pi,$$

subject to:

$$IR: \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f(\pi|e)d\pi - g(e) \geq \overline{u},$$

$$FOIC: \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f_e(\pi|e)d\pi - g'(e) = 0.$$

- To insure that the effort level that satisfies the FOIC condition is indeed a global max we need:

$$\int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi|e))f(\pi|e)d\pi - g(e)$$

to be globally concave in e. Pretty significant restriction.

Adverse Selection, Signaling, and Screening

- Assume productivity is observable. We assume perfectly competitive firms who will hire labor until MRP = MC or:

$$\theta = w.$$

The equilibrium wage of a worker with productivity, θ, is:

$$w^*(\theta) = \theta.$$

- Thus, only workers who are willing to work for wage, $w^*(\theta) = \theta$, are those for whom:

$$r(\theta) \leq w^*(\theta) = \theta.$$

- Profits to the firm are thus $\theta - w = 0$.
- Recall in the case where θ is observable, the firm charges a wage of $w^*(\theta) = \theta$, makes no profits, and only workers of type $\theta \in [r(\theta), \overline{\theta}]$ are employed. This is Pareto optimal since all workers who are better off working (their productivity is higher than their reservation value), are working and those that are better off in their garage, are not employed.
- Suppose now that θ is private information for the worker.
- The set of types of workers who accept to work at wage w is:

$$\Theta(w) = \{\theta | r(\theta) \leq w\}.$$

And the average productivity of those workers is:

$$E[\theta|\theta \in \Theta(w)] = E[\theta|r(\theta) \leq w].$$

- So the equilibrium wage satisfies:

$$w^* = E[\theta|r(\theta) \leq w^*].$$

- Special Case: $r(\theta) = r \in (\underline{\theta}, \overline{\theta})$
- To get pareto optimality, we should get that all workers with $\theta \in [r, \overline{\theta}]$ are employed.

$$E[\theta|r \leq w] = E[\theta].$$

- In equilibrium,
$$w^* = E[\theta],$$

- General Case: $r'(\theta) > 0$. Assume $r(\theta) < \theta \ \forall \ \theta$ and $r(\overline{\theta}) > E[\theta]$. Then the equilibrium wage is:
$$w^* = E[\theta | r(\theta) \leq w^*].$$

- To find the equilibrium outcome, find a fixed point. Denote:
$$\phi(w) = E[\theta | r(\theta) \leq w].$$

- **Remark** Observe that for $r'(\theta) > 0$, $r(\theta) < \theta$ and $r(\overline{\theta}) > E(\theta)$:
$$E[\theta] > E[\theta | r(\theta) \leq w^*].$$

Signaling: Spence

- So consider two types of workers with, $\theta_H > \theta_L > 0$, and let $\lambda = Pr\{\theta = \theta_H\} \in (0, 1)$. Cost, $c(e, \theta)$. Assume the following conditions hold:
  - (1) $c(0, \theta) = 0$. Being a dumbass is free.
  - (2) $c_e(e, \theta) > 0$, $c_{ee}(e, \theta) > 0$. So the cost function is increasing and convex.
  - (3) $c_\theta(e, \theta) < 0$. The high type obtains education with less cost.
  - (4) $c_{e\theta}(e, \theta) < 0$. So the marginal cost of education is decreasing in your productivity. This is called the Spence-Mirrlees Condition, or the "Single Crossing Property". We will discuss this property next time.

- Note that if both a high type and a low type choose the same level of education, this is called a "pooling equilibrium". If they choose different level, we have a "separating equilibrium".

- So a set of strategies (e_L, e_H, and $w(e)$) and the belief function, $\mu(e)$, are a PBE if:
  - (1) The worker's strategy is optimal given the firm's strategies.
  - (2) $\mu(e)$ is derived from the worker's strategies using Bayes Rule to update whenever possible.
  - (3) The firm's wage offers consitute a NE in the simultaneous wage-setting game in which $Pr(\theta_H | e) = \mu(e)$.

- In stage 2, there is Bertrand competition between firms which guarantees:
$$w^*(e) = \mu(e)\theta_H + (1 - \mu(e))\theta_L, \quad (*)$$

Separating Equilibrium (SE)

- We know in a SE that $\mu(e^*(\theta_H)) = 1$ and $\mu(e^*(\theta_L)) = 0$.
- Also, in an SE:
  - (1) In any SE: $w^*(e^*(\theta_H)) = \theta_H$ and $w^*(e^*(\theta_L)) = \theta_L$. This follow directly from equation (*) if we substitute in for $\mu(e)$.
  - (2) In any SE: $e^*(\theta_L) = 0$.
- Let $\underline{e}$ be the minimum level of e for the high type that is consistent with a SE. So set:

$$U_L = \theta_L - c(0, \theta_L) = \theta_H - c(\underline{e}, \theta_L). \quad (**)$$

So conjecture:

$$e^*(\theta_H) = \underline{e}.$$

- So we now have:

$$w^*(e^*(\theta_L)) = \theta_L.$$
$$w^*(e^*(\theta_H)) = \theta_H.$$
$$e^*(\theta_L) = 0.$$
$$e^*(\theta_H) = \underline{e}.$$

- What about off equilibrium path beliefs? We can make these whatever we want as long as they are consistent with a PBE, so consider:

$$\mu(e) = \begin{cases} 0 & \text{for } e < \underline{e} = e^*(\theta_H) \\ 1 & \text{for } e \geq \underline{e} = e^*(\theta_H) \end{cases}$$

And this implies our equilibrium wage schedule:

$$w^*(e) = \begin{cases} \theta_L & \text{for } e < \underline{e} = e^*(\theta_H) \\ \theta_H & \text{for } e \geq \underline{e} = e^*(\theta_H) \end{cases}$$

- **Remark** There exists possibly many SE. Consider $\overline{e}$ to be the highest e consistent with a SE. So it is defined by:

$$U_H = \theta_H - c(\overline{e}, \theta_H) = \theta_L - c(0, \theta_H).$$

So this is the highest threshold level of education that if you pushed the high type any further, they would choose zero education and just get the low wage.

- So for λ sufficiently high, the high type does WORSE than under the no-signaling case. The low type always does WORSE because $\theta_L < E[\theta]$. So we have a rat race where everyone could be doing worse off but no one can afford to deviate. Staying late in the office because everyone else is staying late.

- A key result of the signaling analysis is the requirement of a PBE regarding off the equilibrium-path beliefs.
- One way to restrict the number of equilibria, is to introduce Intuitive Criterion. This means that we eliminate PBE if it is sustained by beliefs that place positive probability on a type for whom a deviation would be equilibrium dominated. That is, the observation of an agent doing an action that if off the equilibrium path wouldn't be reasonable because that agent would have another dominant strategy. This may in fact eliminate ALL equilibria, but it general it is just called an Equilibrium Refinement.

Screening

- Different jobs have different task levels, $t \geq 0$, which do NOT affect the productivity of the worker (or the resulting profits of the firm), but do affect a workers utility:

$$u(w,t,\theta) = w - c(t,\theta),$$

where $c(t,\theta)$ is the amount a worker of type θ suffers when assigned to task t.

- First Best - Types are Public Information
- In equilibrium,

$$t_H^* = t_L^* = 0,$$
$$w_H^* = \theta_H,$$
$$w_L^* = \theta_L.$$

- We have five necessary conditions that characterize the SPE in this private information game:
  - (1) In any SPE, firms must earn zero profits.
  - (2) Any SPE must be a separating equilibrium with $(w_H, t_H) \neq (w_L, t_L)$.
  - (3) In a separating equilibrium, $w_L^* = \theta_L$ and $w_H^* = \theta_H$.
  - (4) In a separating equilibrium, $t_L^* = 0$.
  - (5) In a separating equilibrium, $t_H^* = \hat{t}$ which satisfies:

  $$w_L^* - t_L^* = \theta_L = \theta_H - c(\hat{t}, \theta_L).$$

  So the low type will be just indifferent between accepting $(w_L^*, 0)$ and accepting $(w_H^*, \hat{t}) = (\theta_H, \hat{t})$.
- So, in summary, workers are paid their productivity and $t_L^* = 0$, but:

$$t_H^* > 0.$$

So we have a situation where the task is disorting the otherwise first best equilibrium. High productivity workers are engaging in a costly task only to distinguish themselves from the low types (Loss in welfare).

- Note that a separating equilibrium exists for a low λ, but NOT for a high λ.

26.2 Problem Set Notes

- If $x \sim N(\mu, \sigma^2)$, then $ax \sim N(a\mu, a^2\sigma^2)$ and $e^{ax} \sim$ log normal with,

$$E[e^{ax}] = e^{a\mu + 0.5a^2\sigma^2}.$$

- Suppose we have an agent with initial wealth who invests in a safe asset with return of 1 and risky asset with return $z \sim F(z)$. Suppose he invests y in the risky asset. His final wealth is:

$$\tilde{w} = (w - y) * r + yz = w + y(z - 1).$$

So the problem of the agent:

$$Max_y \int_a^b u(w + y(z - 1))f(z)dz.$$

$$\phi = \int_a^b (z - 1)u'(w + y(z - 1))f(z)dz.$$

Note that sign of dy/dw equals the sign of $d\phi/dw$:

$$\frac{\partial \phi}{\partial w} = \int_a^b (z - 1)u''(w + y(z - 1))f(z)dz = -\int_a^b (z - 1)r_A u'(w + y(z - 1))f(z)dz.$$

26.3 Exams Notes

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