Economics 602: Macroeconomics
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1 Lecture 1: January 26, 2005

1.1 Bob Lucas – "Models of Business Cycles"

- What are the role of recursive tools in economics?
- What are the models that are "useful" for evaluating economic policies?
- First define outputs of a model as both the quantities and the normative outcomes like the overall welfare of the society.
- Usefulness of a model does not involve political motivations but rather we are talking in a more abstract sense.
- Example: The US tax system. Should we change from an income based tax system to one based on consumption?
- We might think that a useful model would involve the idea that economic agents are forward looking. They act in a complex environment when choosing their actions. They have many outlets such as financial instruments. There is also a variety of shocks which can be realized.
- A model that is useful should deal with these complexities. We will now determine useful models for macro policy analysis in a general framework and then look at a specific example (Kydland and Prescott Economy).
- We have two objectives for our models:
  - (1) Fit historical data.
  - (2) Can be simulated "appropriately" to evaluate the outcome of a policy.
- A general model might be:

  $$S_{t+1} = F(S_t, \epsilon_t).$$

  Where S is the state of the economy, a complete description of the state of the macro-economy, ϵ are exogeneous random shocks, so the whole equation is a law of motion for S.
- This general model can be analyzed in two ways:
  - Reduced Form Analysis.
  - Structural Analysis.

Reduced Form Analysis

- Suppose we have data on S, assumptions on the distribution of ϵ, and we go about estimating $F(\cdot)$.

- This would be good for forecasting, but we could not answer our tax policy question because $F(\cdot)$ depends in a complicated way on the components of S, which depend on the actions of the agents and the policy itself. Thus reduced form analysis is not useful for policy analysis, a major limitation.

- Next time we will consider a structural model which considers the behavior of the individual players (components of S).

2 Lecture 2: January 31, 2005

2.1 More on Bob Lucas – "Models of Business Cycles"

- Recall our general law of motion from last time:

$$S_{t+1} = F(S_t, \epsilon_t).$$

- We said reduced form analysis may be ok for forecasting, but otherwise, we needed to do structural analysis which we continue here.

- First we split F into two components: the actions of the agents and the actions of nature. Denote:

$$a_i(s_t), \quad z(s_t),$$

  the actions of agents (household i) and nature at time t, conditional on the state of the economy, s.

- So using this, we have:

$$s_{t+1} = H(z(s_t), a(s_t), s_t, \epsilon_t).$$

  Where a is the aggregate action of all households.

- The main problem with this is embodied in the "Lucas Critique". We cannot separate behavior that is invariant to policy changes and behavior that will change when policy is changed. Thus a should also depend on z but that creates an estimation problem. Thus, structural modelling is not good for policy analysis. $H(\cdot)$ and $a(\cdot)$ vary (structurally) as $z(s_t)$ varies. Thus we move to the next modelling technique.

2.2 Recursive Games - Dynamic Formulation

- Define the set of feasible actions for agents as:

$$a_i \in \Omega_i(a_{-i}, s, z).$$

- Denote instantaneous payoffs (utility, profit, etc) as:

$$R_i(a_t, s_t, z_t).$$

- The objective function of the agent is thus:

$$Max\ E\left[\sum_{t=0}^{\infty} \beta^t R_i(a_t, s_t, z_t)\right]. \quad (1)$$

  We have assumed time separability (strong assumption). We could also say that we are assuming rational expectations, but in this case, just saying that agents all have access to the same information and know the functional forms and the distribution functions, we have rational expectations as a result of our equilibrium concept. We seek a NE in actions of the agents.

- We now wish to write a recursive representation of equation 1. The immediate payoff is $R_i(\cdot)$ and the present discounted value of future payoffs is $\beta E[V_i(s_{t+1})]$. Let $s_{t+1} = s'$, noting that actual timing doesn't matter at this stage, only relative timing. Our Bellman should apply at all dates. Thus the exact form of the B.E. is:

$$V_i(s) = Max_{a_i} \left\{ R_i(a, s, z) + \beta \int V_i(s')dG(\epsilon) \right\},$$

subject to:

$$a_i \in \Omega_i(a_{-i}, s, z).$$

Note that $G(\epsilon)$ is the distribution function of the shocks. Note how we have addressed the Lucas Critique by including z in Ω. There may be aspects of current policy which can affect the feasible set of the agent. We have removed the structural endogeneity.

- The usual problem from here on out is solving for the value function.

- Lucas next provides some basic examples of the recursive approach.

Welfare Effects of Growth and Stabilization Policies

- Assume our instantaneous return function is CRRA so we wish to maximize:

$$Max\ E\left[\sum_{t=0}^{\infty} \beta^t \frac{1}{1-\sigma} c_t^{1-\sigma} \right].$$

Also assume that consumption follows:

$$c_t = (1+\lambda)(1+\mu)^t e^{-0.5\sigma_z^2} z_t,$$

where λ is a scale factor, μ is the mean growth rate of consumption and $log(z_t) \sim N(0, \sigma_z^2)$. Thus,

$$E[c_t] = (1+\lambda)(1+\mu)^t.$$

- We need to fit the model to the data. Note that US post war growth has been about 3 percent and consumption variation is around 1.3 standard deviations. Thus set:

$$[\lambda, \mu, \sigma_z^2] = [0, 0.03, 0.013^2].$$

- Next time we will look at the compensating variation (CV) for some policy changes. In particular, we will try to determine what change in λ (the shift parameter) is necessary to make the agent just as well off with and without a change in the mean consumption growth (μ) and the variability of consumption (σ_z^2).

3 Lecture 3: February 2, 2005

3.1 More on Lucas, "Models of Business Cycles"

Example 1: The Welfare Effects of μ and σ_z^2

- Recall we assumed preferences were CRRA:

$$U(c) = E[\sum \beta^t \frac{1}{1-\sigma} c_t^{1-\sigma}],$$

and the consumption function of the representative agent was:

$$c_t = (1+\lambda)(1+\mu)^t e^{-0.5\sigma_z^2} z_t, \quad ln(z_t) \sim N(0,1).$$

- Fitting to the data:

$$[\lambda, \mu, \sigma_z^2] = [0, 0.03, (0.013)^2].$$

- Since consumption depends on the parameters of the model, we also know the indirect utility function depends on those parameters. Since utility is an ordinal concept, we must map it into something quantifiable, like the percent change in consumption, which we measure through changes in λ, the shift parameter of the model.

- Consider a Growth Policy (μ).

- We look for the percent change in consumption at all dates and in all states that leaves the agent indifferent between an economy growing at rate μ and one growing at rate μ_0. Thus we equate the indirect utility at growth rate μ and shift parameter level $F(\mu, \mu_0)$ to the indirect utility at growth rate μ_0 and shift parameter level 0. Thus,

$$\begin{aligned}
U(F(\mu,\mu_0), \mu, \sigma_z^2) &= U(0, \mu_0, \sigma_2^z) \\
E\left[\sum \beta^t \frac{((1+\lambda)(1+\mu)^t e^{-0.5\sigma_z^2} z_t)^{1-\sigma}}{1-\sigma}\right] &= E\left[\sum \beta^t \frac{((1+\mu_0)^t e^{-0.5\sigma_z^2} z_t)^{1-\sigma}}{1-\sigma}\right] \\
E\left[\sum \beta^t ((1+\lambda)(1+\mu)^t e^{-0.5\sigma_z^2} z_t)^{1-\sigma}\right] &= E\left[\sum \beta^t ((1+\mu_0)^t e^{-0.5\sigma_z^2} z_t)^{1-\sigma}\right] \\
& \quad Somehow* \\
\lambda = F(\mu, \mu_0) &= \left(\frac{1+\mu_0}{1+\mu}\right)^{\beta/(1-\beta)} - 1
\end{aligned}$$

*See Fabiano's notes for an almost correct derivation. Note that the variance of consumption does not come into play here. Lucas goes on to show the effects of policy changes that effect growth result in a huge impact on welfare.

- Consider a Stabilization Policy (σ_z^2).

- We look for the percentage change in consumption at all dates and in all states that leaves the agent indifferent between an economy with no fluctuations in the business cycle and one with fluctuations, σ_z^2.

$$\begin{aligned} U(g(\sigma_z^2)), \mu, \sigma_z^2) &\quad = \quad U(0, \mu, 0) \\ &\textit{Somehow}* \\ \lambda = g(\sigma_z^2) &\quad \approx \quad \frac{1}{2}\sigma\sigma_z^2 \end{aligned}$$

*See Fabiano's notes. Lucas shows the effects of stabilization policy are very small compared to growth policy, though they depend on the risk aversion parameter, σ.

- Key Quote: "The policies that pursue stabilization have a much smaller impact on welfare than the policies that pursue growth."
- Criticisms of the Lucas paper. The model may not be very realistic given the representative agent, the full insurance assumption that is implicit, and the possible connection between growth and stabilization (where here we treat them separately). We also might ask, do growth policies really effect growth? The result of all this might be that recessions do NOT matter ... could this be true? The effects of the stabilization policy also depend on the type of economy we are considering. If the variability is already low, the effects will be small where in say a developing nation, the effects could be amplified.

Example 2: Kydland - Prescott Model

- 4 main characteristics of the model:
  - (1) Representative agent.
  - (2) Perfectly competitive closed economy.
  - (3) 1 good produced with a CRS production function using inputs, K and L.
  - (4) There are total factor productivity (TFP) shocks present.
- The Model. Preferences:

$$E[\sum \beta^t U(c_t, \overline{n} - n_t)],$$

where $\overline{n}$ is the agent's total time endowment and n_t is labor hours worked. Technology:

$$F(K_t, n_t, x_t),$$

where x_t is the TFP shock which follows a first order markov process. Recall the markov property:

$$G(x', x) = Pr\{x_{t+1} \leq x' | x_t = x\}.$$

Finally, the capital evolution equation is:

$$K' = i + (1 - \delta)K.$$

4 Lecture 4: February 7, 2005

4.1 More on Lucas, "Models of Business Cycles"

Kydland/Prescott Economy

- Recall we have an atomistic representative firm and consumer who takes prices and rental rates as given. Preferences are denoted:

  $$E[\sum_{t=0}^{\infty} \beta^t u(c_t, \overline{n} - n_t)].$$

  Technology:

  $$F(K_t, n_t, x_t),$$

  where x_t follows a first order markov process with distribution, $G(x_{t+1}, x_t)$. Capital evolves according to:

  $$K_{t+1} = (1 - \delta)K_t + i_t.$$

- So consider the following elements of the problem:
  - (a) State of the system at date t.
    - $y \equiv$ agent i's holdings of capital.
    - $K \equiv$ aggregate capital stock (exogeneous to the representative agent).
    - $x \equiv$ technology shocks (also exogeneous).

    So the state vector is:

    $$s = (y, K, x),$$

    where (y) is a state specific to the agent and (K, x) are exogeneous aggregate states.
  - (b) Actions. $a_i = a$ with a a function of $a(c, n, y')$. The agent chooses consumption, labor supplied, and next period's capital (though there will be redudancies).
  - (c) Immediate payoff. $R(a) = u(c, \overline{n} - n)$, the instantaneous felicity function.
  - (d) Feasible actions. These depend on aggregate variables outside the control of our agent.

    $$\Omega(y, K, x) = \{(c, n, y') | c + y' \leq w(K, x) * n + u(K, x) * y + (1 - \delta)y;$$

    $$c \geq 0, 0 \leq n \leq \overline{n}, y' - (1 - \delta)y \geq 0\}.$$

    Where $w(K, x)$ and $u(K, x)$ are the wage and rental rate of capital which depend on the aggregate states.
  - (e) Aggregate law of motion. $K' = (1 - \delta)K + i$, or more generally, $K' = h(K, x)$.

- Individual's Optimal Plans:

$$V(s) = Max \ \ \{u(c, \overline{n} - n) + \beta \int v(y', \underbrace{h(K, x)}_{K'}, x') dG(x', x)\},$$

subject to:

$$(c, n, y') \in \Omega.$$

- The solution will look like:

$$y' = y(y, K, x),$$

a recursive competitive equilibrium.

- In equilibrium, we expect that:

$$y(K, K, x) = h(K, x)$$

identically in (K, x). This means that the policy function of the individual agent (by solving for a Nash equilibrium) will look the same as that of say a social planner who solves for the optimal social equilibrium. Since $y(\cdot)$ is the choice of the agent and $h(K, x)$ is the agent's conjecture about how the aggregate capital stock evolves, it makes sense that in a Nash equilibrium, these should be the same.

- The social planner's problem can be written (by appealing to the Welfare theorems):

$$f(K, x) = Max_{(c,n,K')} \ \ \{u(c, \overline{n} - n) + \beta \int f(K', x') dG(x'x)\},$$

subject to:

$$K' + c \leq F(K, n, x) + (1 - \delta)K, c \geq 0, 0 \leq n \leq \overline{n}, K' - (1 - \delta)K \geq 0.$$

4.2 Principals of Dynamic Programming

General Intertemporal Problem

- Denote: x_t, a $nx1$ vector of state variables for $t = 0, \ldots, T + 1$.
- Denote: u_t, a $kx1$ vector of choice variables for $t = 0, \ldots, T$.
- Problem: Choose $\{x_t\}_{t=1}^{T+1}$ and $\{u_t\}_{t=0}^{T}$ to:

$$Max \ \{R(x_0, u_0, x_1, u_1, \ldots, x_T, u_T, x_{T+1})\},$$

such that:

$$x_0 \ given,$$

$$\text{Constraints: } G(x_0, u_0, x_1, u_1, \ldots, x_T, u_T, x_{T+1}) \geq 0.$$

Where $G(\cdot)$ is a set of $(T + 1) * n$ functions.

- We could then write the lagrangian as:

$$\mathcal{L} = R(\cdot) + \mu' G(\cdot),$$

where μ is a $[(T+1)n]x1$ vector of lagrange multipliers, one for each constraint.

- But we would probably have trouble solving this problem. Hence we will impose structure (make assumptions) to make the problem more tractable. We would like the problem to be "Recursive in Time" or in other words, we need "Time Separability." The constraints and payoff function must be time separable.

Time Recursive Intertemporal Problem

- Assumptions:
  - (1) $R(\cdot)$ is time separable into period payoffs, $r_t(x_t, u_t)$, where r_t is C^2 and concave.
  - (2) $\omega = \{x_{t+1}, x_t, u_t : x_{t+1} \leq g_t(x_t, u_t), u_t \in \Re^K\}$ is convex and compact.
- Problem is now:

$$Max\ \{\sum_{t=0}^{T} r_t(x_t, u_t) + W_0(x_{T+1})\},$$

subject to:

$$x_0,$$
$$x_{t+1} = g_t(x_t, u_t)\ for\ t = 0, \ldots, T,$$

where $W_0(\cdot)$ is the terminal payoff.

- We could again write out the lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T} r_t(x_t, u_t) + W_0(x_{T+1}) + \lambda_0'[g_0(x_0, u_0) - x_1] + \cdots + \lambda_T'[g_T(x_T, u_T) - x_{T+1}]. \quad (1)$$

FOCS:

  - [1.1] $\dfrac{\partial \mathcal{L}}{\partial u_t} = \dfrac{\partial r_t}{\partial u_t} + \dfrac{\partial g_t}{\partial u_t}\lambda_t = 0$ for $t = 0, \ldots, T$.
  - [1.2] $\dfrac{\partial \mathcal{L}}{\partial x_t} = \dfrac{\partial r_t}{\partial x_t} + \dfrac{\partial g_t}{\partial x_t}\lambda_t - \lambda_{t-1} = 0$ for $t = 1, \ldots, T$.
  - [1.3] $\dfrac{\partial \mathcal{L}}{\partial x_{T+1}} = W_0'(x_{t+1}) - \lambda_T = 0$.
  - [1.4] $x_{t+1} = g_t(x_t, u_t)\ for\ t = 0, \ldots, T$.

5 Lecture 5: February 9, 2005

5.1 Dynamic Programming

Manipulate FOCs

- Recall the set of FOCs from last time for our time separable problem:
  - [1.1] $\frac{\partial \mathcal{L}}{\partial u_t} = \frac{\partial r_t}{\partial u_t} + \frac{\partial g_t}{\partial u_t}\lambda_t = 0$ for $t = 0, \ldots, T$.
  - [1.2] $\frac{\partial \mathcal{L}}{\partial x_t} = \frac{\partial r_t}{\partial x_t} + \frac{\partial g_t}{\partial x_t}\lambda_t - \lambda_{t-1} = 0$ for $t = 1, \ldots, T$.
  - [1.3] $\frac{\partial \mathcal{L}}{\partial x_{T+1}} = W_0'(x_{t+1}) - \lambda_T = 0$.
  - [1.4] $x_{t+1} = g_t(x_t, u_t)\ for\ t = 0, \ldots, T$.
- To solve the system, you need to take advantage of the recursive nature of the problem. Solve (1.2) for the multiplier and update:

$$\lambda_t = \frac{\partial r_{t+1}}{\partial x_{t+1}} + \frac{\partial g_{t+1}}{\partial x_{t+1}}\lambda_{t+1}.\ \ (*)$$

  This is like an euler equation for λ.

- Now consider rewriting equation 1.1 (at time T) and substituting in (1.3):

$$\phi_T^1 = \frac{\partial r_T}{\partial u_T} + \frac{\partial g_T}{\partial u_T}W_0'(x_{T+1}) = 0.$$

  And now update 1.4 to time T:

$$\phi_T^2 = x_{T+1} - g_T(x_T, u_T) = 0.$$

  These two equations describe the system of FOCs at date T.

- Now move to date $T-1$:

$$\phi_{T-1}^1 = \frac{\partial r_{T-1}}{\partial u_{T-1}} + \frac{\partial g_{T-1}}{\partial u_{T-1}}\lambda_{T-1} = 0.$$

  Substitute from (*):

$$\phi_{T-1}^1 = \frac{\partial r_{T-1}}{\partial u_{T-1}} + \frac{\partial g_{T-1}}{\partial u_{T-1}}[\frac{\partial r_T}{\partial x_T} + \frac{\partial g_T}{\partial x_T}\lambda_T] = 0.$$

  Substitute from 1.3:

$$\phi_{T-1}^1 = \frac{\partial r_{T-1}}{\partial u_{T-1}} + \frac{\partial g_{T-1}}{\partial u_{T-1}}[\frac{\partial r_T}{\partial x_T} + \frac{\partial g_T}{\partial x_T}W_0'(x_{T+1})] = 0.$$

And again for ϕ^2:

$$\phi^2_{T-1} = x_T - g_{T-1}(x_{T-1}, u_{T-1}) = 0.$$

- We could continue this and form our ϕ equations at any time t:

$$\phi^1_t = \frac{\partial r_t}{\partial u_t} + \frac{\partial g_t}{\partial u_t}[\frac{\partial r_{t+1}}{\partial x_{t+1}} + \cdots + \frac{\partial g_T}{\partial x_T} W'_0(x_{T+1})] = 0.$$

$$\phi^2_t = x_{t+1} - g_t(x_t, u_t) = 0.$$

- Even with these general equations, this is still a hard system to solve. However, we have no multipliers in the equations and the system is clearly recursive with each set of ϕ's depending only on current and forward dated variables, nothing from the past. This lends itself to a strategy of solving the system backwards.

Backward Recursive Strategy

- Rewrite the ϕ's at time T as:

$$\phi^1_T(\overline{x}_T, u_T, x_{T+1}) = 0.$$

$$\phi^2_T(\overline{x}_T, u_T, x_{T+1}) = 0.$$

So since $\overline{x}_T$ is fixed (given), we could solve for:

$$u_T = h_T(x_T)$$

$$x_{T+1} = g_T(x_T, \underbrace{h_T(x_T)}_{u_T}) = f_T(x_T).$$

- Now at time $T-1$,

$$\phi^1_{T-1}(\overline{x}_{T-1}, u_{T-1}, x_T) = 0.$$
$$\phi^2_{T-1}(\overline{x}_{T-1}, u_{T-1}, x_T) = 0.$$

So since $\overline{x}_{T-1}$ is fixed (given), we could solve for:

$$u_{T-1} = h_{T-1}(x_{T-1})$$

$$x_T = g_{T-1}(x_{T-1}, h_{T-1}(x_{T-1})) = f_{T-1}(x_{T-1}).$$

- And so on until in general we have:

$$u_{t-1} = h_{t-1}(x_{t-1})$$

$$x_t = f_{t-1}(x_{t-1}).$$

And these are our feedback rules at any point in time t.

- Also imbedding in all of this is time consistency. The plans choosen by agents at time $t = s$ must be the same that are choose at time $t = s + 1$. There is no incentive to deviate from one's plan at some point in the future. This is exactly "Bellman's Principal of Optimality."

Bellman's Equation

- Another way to characterize the solution to the backward recursion strategy is using a Bellman's equation.

- Define the 1-period, date T, value function:

$$W_1(x_T) = Max_{u_T} \{r_T(x_T, u_T) + W_0(x_{T+1})\},$$

subject to:

$$x_{T+1} = g_T(x_T, u_T) \equiv \phi_T^2,$$

$$x_T \text{ given.}$$

- FOC of this problem:

$$\frac{\partial r_T}{\partial u_T} + W_0'(x_{T+1})\frac{\partial g_T}{\partial u_T} = 0 \equiv \phi_T^1.$$

- Envelope condition (evaluated at the optimal $u_T = h_T(x_T)$):

$$W_1'(x_T) = \frac{\partial r_T(x_T, h_T(x_T))}{\partial x_T} + \frac{\partial g_T(x_T, h_T(x_T))}{\partial x_T} W_0'(g_T(x_T, h_T(x_T))).$$

- Next define the 2-period, date $T-1$, value function:

$$W_2(x_{T-1}) = Max_{u_{T-1}} \{r_{T-1}(x_{T-1}, u_{T-1}) + W_1(x_T)\},$$

subject to:

$$x_T = g_{T-1}(x_{T-1}, u_{T-1}) \equiv \phi_{T-1}^2,$$

$$x_{T-1} \text{ given.}$$

- FOC of this problem:

$$\frac{\partial r_{T-1}}{\partial u_{T-1}} + W_1'(x_T)\frac{\partial g_{T-1}}{\partial u_{T-1}} = 0.$$

Note this isn't quite equivalent to ϕ_{T-1}^1. Substitute in the date T envelope condition though and you will get ϕ_{T-1}^1 exactly.

- We could repeat this, so in general, the $j+1$ period, date $T-j$, value function is:

$$W_{j+1}(x_{T-j}) = Max_{u_{T-j}} \{r_{T-j}(x_{T-j}, u_{T-j}) + W_j(x_{T-j+1})\},$$

subject to:

$$x_{T-j+1} = g_{T-j}(x_{T-j}, u_{T-j}) \equiv \phi_{T-j}^2,$$

$$x_{T-j} \text{ given.}$$

- FOC of this problem will be the same as before (with appropriate time subscripts) and the envelope result would give us:

$$W'_{j+1}(x_{T-j}) = \frac{\partial r_{T-j}(x_{T-j}, h_{T-j}(x_{T-j}))}{\partial x_{T-j}} + \frac{\partial g_{T-j}(x_{T-j}, h_{T-j}(x_{T-j}))}{\partial x_{T-j}} W'_j(g_{T-j}(x_{T-j}, h_{T-j}(x_{T-j}))).$$

- Finally, note that the W'_j term is like a lagrange multiplier, the shadow value of the constraint. In a RBC setting where W might be wealth, the term is like the marginal utility of wealth or the shadow value of the budget constraint.

- We will talk later how the whole notion of time-INconsistency, often studied in macroeconomic settings, is really just the breakdown of Bellman's Principal of Optimality.

6 Lecture 6: February 14, 2005

6.1 Dynamic Programming

- Bellman's Optimality Principal: All the policies are self-inforcing. We can break down the problem to a day zero maximization problem insteady of solving for each u_t. The original plan is unchanged if the individual is allowed to revise his decision every period. Thus there is an incentive not to deviate. This is very similar to time consistency.

Discounted Dynamic Programming

- We start with the following assumptions:
  - (1) The functional form is time independent.

$$r_t(x_t, u_t) = \beta^t r(x_t, u_t), \quad \beta \in [0, 1].$$

  - (2) Time invariant constraints:

$$g_t(x_t, u_t) = g(x_t, u_t).$$

- What do we gain with these assumptions? We want to consider discounted DP because in a general problem, the value functions converge, so we can prove we get time invariant solutions where the optimal control and states evolve according to:

$$u_t = h(x_t).$$

$$x_{t+1} = f(x_t).$$

Note that the h and f functions do not have time subscripts.

- The functions will converge to true invariant functions.
- So in general, the $j+1$ period, date $T-j$, discounted value function is:

$$\beta^{j-T} W_{j+1}(x_{T-j}) = Max_{u_{T-j}} \{\beta^{T-j} r(x_{T-j}, u_{T-j}) + W_j(x_{T-j+1})\},$$

subject to:

$$x_{T-j+1} = g(x_{T-j}, u_{T-j}),$$

$$x_{T-j} \text{ given.}$$

Here W is our present-value value function. We can also write our current-value value function as:

$$V_{j+1}(X_{T-j}) = \beta^{j-T} W_{j+1}(x_{T-j}).$$

And $V_{T+1}(x_0) = W_{T+1}(x_0)$.

- We want to make sure the final time period is still time invariant and to preserve the time invariant nature. Hence we present the problem:

$$V_{j+1}(x) = Max_u\{r(x,u) + \beta V_j(x')\},$$

subject to:

$$x' = g(x,u),$$

$$x \text{ given.}$$

- So far we haven't proved that our decision rule and value functions are time invariant. Will iterations on the value function starting from any V_0 converge to the "true" value function? Ie,

$$V = lim_{j\to\infty} V_j$$

- We must make a distinction between finite and infinte time. For finite time period models, we'll get a time variant value function. When $T \to \infty$, as we will do below, we may find that the value function is time invariant.

- So when do the iterations converge? When the iteration follows a Contraction Mapping that satisfies Blackwell's Sufficient Conditions (BSCs).

- Key Assumptions.
  - (1) Convergency. $r(x,u)$ is real-valued, continuous, concave and bounded. (Might represent a problem for CRRA utility).
  - (2) The set:

    $$[x', x, u : x' \leq g(x,u), u \in \Re^K]$$

    is convex and compact.

- **Definition:** Define a contraction operator, T, for iterations on $V_j(x)$:

$$V_1 = TV_0 = Max\ \{r(x,u) + \beta V_0(x')\}.$$

So, we're mapping the $r(\cdot)$ function that satisfies the above assumptions into another function that is real-valued, continuous and bounded. We could also prove that we are mapping in a concave function. What we really need is a fixed point condition.

- How do we measure convergence? Consider:

$$d_\infty(V,W) = sup_{x\in X}\ |V_{j+1}(x) - V_j(x)| \approx \epsilon.$$

Convergence comes from allowing ϵ to go to zero.

- To get uniqueness, we need BSCs. These are:
  - (1) Monotonicity: $V_{j+1}(x) \geq V_j(x) \Rightarrow TV_{j+1}(x) \geq TV_j(x)$ so,

    $$TV_{j+2}(x) \geq TV_{j+1}(x).$$

– (2) Discounting:

$$T(V_j(x) + c) = TV_j(x) + \beta c.$$

- If these conditions are satisfied, then T is a contraction mapping on a complete metric space. Then we know that T has a unique fixed point.

- Using Blackwell's fixed point theorem, when $V_{j+1} = TV_j$ is a contraction mapping on a complete metric space (ie, monotonicity and discounting are satisfied), then there is a unique V that solves $V = TV$.

- Note that BSCs are just sufficient. We can use other ways to prove a fixed point exists. In the case of CRRA utility, we cannot prove it through BSCs. For precautionary savings, we cannot prove the existence unless we impose that the interest rate is less than the discount rate.

- Also note that not all fixed points are a contraction mapping. We could have a fixed point that if you're just a bit away from it, it spirals out.

7 Lecture 7: February 16, 2005

7.1 Dynamic Programming and Examples

- Properties of the limiting value function:

$$V(x) = Max\ \{r(x,u) + \beta V(x')\},$$

subject to:

$$x' = g(x,u), \quad x \text{ given.}$$

  - (1) $V(\cdot)$ is unique and strictly concave.
  - (2) $V(\cdot)$ is approached as $lim_{j\to\infty} V_j$ from iterations on the Bellman starting from any bounded continuous V_0.
  - (3) $h(x_t)$ and $f(x_t)$ are unique and time invariant policy functions or decision rules.
  - (4) $V(\cdot)$ is differentiable with:

$$V'(x) = \frac{\partial r(x,h(x))}{\partial x} + \beta \frac{\partial h(x,h(x))}{\partial x} V'(g(x,h(x))).$$

Example - Cass-Koopmans Optimal Growth

- Consider the planner's problem:

$$Max \sum_{t=0}^{\infty} \beta^t u(c_t),$$

subject to:

$$c_t + k_{t+1} = f(k_t).$$

- Assume: $u'(0) = \infty, u' > 0, u'' < 0$ and $f'(0) = \infty, f'(\infty) = 0, f' > 0, f'' < 0$.
- Our state variable is k and our choices are c and k'. Thus, the Bellman:

$$V(k) = Max\ \{u(c) + \beta V(k')\},$$

subject to:

$$c + k' = f(k).$$

- FOC:

$$u'(f(k) - k') = \beta V'(k') \Longrightarrow \underbrace{k' = h(k)}_{policy\ function}.$$

- Envelope:

$$V'(k) = u'(f(k) - h(k))f'(k).$$

Updated:

$$V'(k') = u'(f(k') - h(k'))f'(k').$$

Substitute:

$$u'(c) = \beta u'(c')f'(k').$$

Or,

$$\frac{u'(c)}{\beta u'(c')} = f'(k').$$

Which says the intertemporal marginal rate of substitution should equal the interest rate on capital. This is like the tangency between an indifference curve and the production frontier.

- Properties of the Cass-Koopman's model:
  - (1) $h(k)$ is non-decreasing in k and $V(k)$ is strictly concave.
  - (2) There exists a maximum stationary capital stock. With $c_t = 0 \ \forall \ t$, $k_{t+1} = f(k_t)$ and this has a fixed point when $\overline{k} = f(\overline{k})$. So there is an interval, $(0, \overline{k}]$ where the equilibrium exists because since people like consumption, $c_t > 0$. See G-7.1.
  - (3) For any $k_0 \in (0, \overline{k}]$, an equilibrium, $k_t \in (0, \overline{k}] \ \forall \ t$.
  - (4) $\{k_t\}_{t=0}^{\infty}$ is a monotone bounded sequence so it must converge. Since $h(k)$ is non-decreasing, capital must be growing at every time period. So $\{k_t\}$ is a monotone bounded sequence which converges to the limit or steady state, $k_\infty(k_0)$ as $t \rightarrow \infty$.
  - (5) k_∞ is independent of initial conditions (no matter which level of capital we start at, we always converge).

- So what is this steady state? It's when the intertemporal marginal rate of substitution equals the real interest rate. So, from our Euler:

$$u'(c_\infty) = \beta u'(c_\infty)f'(k_\infty).$$

$$\frac{1}{\beta} = f'(k_\infty).$$

So the rate of time preference must equal the real interest rate. This is our unique steady state.

Example - Brock-Mirman

- Here, we have the same setup as in Cass-Koopman's except $u(c) = log(c)$ and $f(k) = Ak^\alpha$.

- This problem has a closed form solution (which we can see via guess and verify):

$$V(k) = E + Fln(k).$$

7.2 Dynamic Programming with Uncertainty

- We need to characterize dynamics in a discrete space (random variable) so we will use a Markov Process.

- **Definition:** $\{x_t\}$ satisfies the Markov Property if:

$$Pr(x_{t+1}|x_t, x_{t-1}, \ldots, x_{t-k}) = Pr(x_{t+1}|x_t)\ \forall\ t, k \geq 2.$$

- A Markov process is characterized by a Markov chain which has three elements:
  - (1) Vector realizations: $\overline{x} \in \Re^n$.
  - (2) Initial probabilities: π_0, an $nx1$ vector.
  - (3) Transition matrix, P, an nxn matrix.

- P and π_0 are stochastic objects so the elements of π_0 sum to 1 and the rows of P sum to 1. All elements are non-negative.

- Facts about markov chains:
  - (1) $\pi_1' = \pi_0' P$, $\pi_2' = \pi_0' P^2$, or in general $\pi_{t+1}' = \pi_t' P$.
  - (2) We say there is a fixed point on π if there exists an invariant distribution (also called ergodic, limiting, or stationary distribution) such that:

$$\pi' = \pi' P.$$

    Or,

$$(I - P')\pi = 0.$$

    Thus we look for the eigenvalues of P.

- Simple example 1. Consider the transition matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0.2 & 0.5 & 0.3 \\ 0 & 0 & 1 \end{bmatrix}.$$

  This matrix has 2 unit eigenvalues:

$$\pi' = (1, 0, 0), \text{ and } \pi' = (0, 0, 1).$$

  This is intuitive since states 1 and 3 are both absorbant states. We could never end up in state 2 in the limiting distribution.

- Simple example 2. Consider the transition matrix:

$$P = \begin{bmatrix} 0.7 & 0.3 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.9 & 0.1 \end{bmatrix}.$$

This matrix has 1 unit eigenvalue:

$$\pi' = (0, 0.64, 0.36).$$

This is intuitive since we cannot end up in state 1 because once we are in state 2 or 3, we can't get back to state 1.

- Fact: If P has all strictly positive elements, then it has a single unique invariant distribution.
- **Definition:** A markov process is asymptotically stationary with a unique invariant distribution if:
  - (1) It has a unique stationary distribution that solves $(I - P')\pi_\infty = 0$.
  - (2) For all π_0's, $lim_{t\to\infty}\pi_0' P^t = \pi_\infty$.

8 Lecture 8: February 21, 2005

8.1 More on Markov Processes

- **Definition:** A Markov process is asymptotically stationary (AS) with a unique invariant distribution if:
  - (1) If it has a unique, π_∞, such that, $(I - P')\pi_\infty = 0$.
  - (2) If for all π_0, $lim_{t\to\infty} p^{t'}\pi_0 = \pi_\infty$.
- **Theorem:** Existence. Consider a Markov process with transition matrix, P. Then:
  - (1) If all elements of P, $P_{ij} > 0$, then P has a unique, invariant π_∞, and is AS.
  - (2) If for any $n \geq 1$, all elements of P^n, $P^n_{ij} > 0$, then P has a unique, invariant π_∞, and is AS.
- How do we compute π_∞? One of three ways:
  - (1) Solve $(I - P')\pi_\infty = 0$. This works if P is small.
  - (2) Iterate $P^{t'}\pi_0$ until it converges. This is the most practical in most cases.
  - (3) Power P so that all its rows converge.
- **Definition:** Expectations:

$$\text{Unconditional Expectation: } E[x_t] = (\pi_0' P^t)\overline{x}.$$

$$\text{Conditional Expectation: } E[x_{t+k}|x_t = \overline{x}] = P^k\overline{x}.$$

- Often we are not interested in the actual Markov process, x, but rather some function of x. For example, capital might follow a Markov process and GDP is a function of capital via Cobb-Douglas technology.
- **Definition:** Forecasting Functions of Markov Processes. Define $h(\overline{x})$ to be our forecasting function. $h(\cdot)$ need not be linear. Then:
  - (1) $E[h(x_{t+k})|x_t = \overline{x}] = P^k h(\overline{x})$.
  - (2) $E[\sum_{k=0}^{\infty}\beta^k h(x_{t+k})|x_t = \overline{x}] = (I - \beta P)^{-1}h(\overline{x})$.

8.2 Stochastic Discounted Dynamic Programming

- Consider a model of an economy with payoff function:

$$E_0[\sum_{t=0}^{\infty}\beta^t r(x_t, u_t)], \text{ subject to: } x_{t+1} = g(x_t, u_t, \epsilon_t), \ x_0 \text{ given.}$$

  Note the timing of ϵ_t may be adjusted depending on when decisions are made. Assume:

$$\{\epsilon_t\}_{t=0}^{\infty}$$

is either continuous *iid* or Markov. If it is *iid*, we need to know the distribution and whatever moments define that distribution. If it is Markov, we need the three elements: $(P, \pi_0, \overline{x})$.

- Note that we can translate a time series process (VAR) into a Markov chain using quadratic methods. See Taucheu, Econometrica, 1991.

- The solutions to the above problem will be functions:

$$u_t = h(x_t, \epsilon_t), \text{ and } x_{t+1} = f(x_t, \epsilon_t).$$

- Note that x_t is called the endogeneous state and ϵ_t is an exogeneous state driven by some dynamics outside of the model. Suppose ϵ_t is Markov with a $2x2$ transition matrix so that $\epsilon_t \in [\epsilon^H, \epsilon^L]$. Suppose that x_t takes on one of NX possible values. Then we need more than just the transition matrix of ϵ, we need the joint transition matrix of x with epsilon. The joint matrix would be of size $(NX * 4)x(NX * 4)$. Suppose we start out at ϵ^L and x_1. What is the probability that we end up at ϵ^L and x_2? Well, if the feedback rule, $f(x_t, \epsilon_t)$ is unique, then for any given x_t and ϵ_t, we know exactly x_{t+1}. Suppose $f(x_1, \epsilon^L) = x_4$. Then:

$$Prob\{(x_t, \epsilon_t) = (x_2, \epsilon^L)|(x_1, \epsilon^L)\} = 0.$$

And,

$$Prob\{(x_t, \epsilon_t) = (x_4, \epsilon^L)|(x_1, \epsilon^L)\} = Prob\{\epsilon^L|\epsilon^L\}.$$

Also,

$$Prob\{(x_t, \epsilon_t) = (x_4, \epsilon^H)|(x_1, \epsilon^L)\} = Prob\{\epsilon^H|\epsilon^L\}.$$

So this is how we populate the transition matrix for the endogeneous and exogeneous states. It is clear that most of the entries will be zero. We can however appeal to the second theorem above to (hopefully) show that the process still has a unique, invariant, and AS distribution. See Lucas and Stokey.

- So what's the Bellman's Equation for our problem:

$$V(x, \epsilon) = Max_u \ \{r(x, u) + \beta E[V(x', \epsilon')|x, \epsilon]\}, \text{ subject to : } x' = g(x, u, \epsilon), \ x_0 \text{ given.}$$

$$V(x, \epsilon) = Max_u \ \{r(x, u) + \beta E[V(g(x, u, \epsilon), \epsilon')|x, \epsilon]\}.$$

So we can start with any bounded V_0, and iterate until we find the value function.

Example: Brock Mirman, 1972 - Stochastic Growth Problem

- Consider the problem

$$Max \ E_0[\sum_{t=0}^{\infty} \beta^t log(c_t)],$$

subject to:

$$c_t + k_{t+1} = Ak_t^{\alpha}\theta_t, \text{ where } log(\theta_t) \sim \ iid \ N(0, \sigma^2).$$

So θ_t is an *iid* shock to TFP.

- Bellman:
$$V(\theta,k) = Max_{k'}\ \{log(c) + \beta E[V(k',\theta')|k,\theta]\},$$
such that:
$$k' = Ak^{\alpha}\theta - c.$$

- It can be shown via guess and verify that $V(\theta,k) = E + F\ log(k) + G\ log(\theta)$. This induces policy functions:
$$k' = \alpha\beta Ak^{\alpha}\theta.$$
$$c = (1-\alpha\beta)Ak^{\alpha}\theta.$$

- Recall the deterministic steady state (Cass/Koopmans Model) was:
$$k = (\alpha\beta A)^{1/(1-\alpha)}.$$
$$log(k) = \frac{log(\alpha\beta A)}{1-\alpha}.$$

- Our average stochastic capital stock is derived as:
$$log(k') = log(\alpha\beta A) + \alpha log(k) + log(\theta).$$
$$log(k) = log(\alpha\beta A) + \alpha log(k) + log(\theta).$$
$$E[log(k)] = log(\alpha\beta A) + \alpha E[log(k)].$$
$$E[log(k)] = \frac{log(\alpha\beta A)}{1-\alpha}.$$
So the expectation of the stochastic capital stock is the same as in the deterministic model.

9 Lecture 9: February 23, 2005

9.1 The Time Inconsistency Problem - Calvo '78

- Consider the problem of private agents and the government.
  - (1) Private Agents:

$$Max_{\{c_t, M_{t+1}\}} \sum \beta^t u(c_t, \frac{M_{t+1}}{p_t}),$$

subject to:

$$c_t + \tau_t + \frac{M_{t+1}}{p_t} = y(\tau_t) + \frac{M_t}{p_t},$$

$$M_0 \; given,$$

$$u(\cdot) = ln(c_t) + \gamma \; ln(\frac{M_{t+1}}{p_t}).$$

So the agent maximizes over consumption and money balances subject to his consumption plus tax payments plus next period money holdings being less than his income, as a function of taxes, plus current money balances. Note we are not making the tax scheme explicitly distortationary and we are not saying why the consumer likes to hold cash. Just assumptions of the model.

  - (2) Government:

$$Max_{\{M_{t+1}, \tau_t\}} \sum \beta_t u(\underbrace{c_t^*, \frac{M_{t+1}^*}{p_t}}_{optimal}),$$

subject to:

$$c_t + g_t = y(\tau_t),$$

$$g_t = \tau_t + \underbrace{\frac{M_{t+1} - M_t}{p_t}}_{seigniorage}.$$

- So an equilibrium is characterised by allocations, $\{c_t, M_{t+1}, \tau_t\}$ and prices $\{p_t\}$ such that private agents maximize their utility subject to their constraints taking $\{p_t, \tau_t\}$ as given. The government also maximizes its payoff function taking $\{p_t\}$ as given but taking into account the private agent's optimized $\{c_t^*, M_{t+1}^*\}$.

- Note that g, government spending is NOT choosen, but assumed to be exogeneous and does not enter either agent's payoff function directly.

- So the private agent's dynamic programming problem is as follows. τ and m are both states so let:

$$x \equiv y(\tau) + \frac{M}{p}$$

be our state variable, the private agent's income at each period. Thus, our Bellman's Equation is:

$$V(x) = Max_{c,M'} \{u(c, \frac{M'}{p}) + \beta V(y(\tau') + \frac{M'}{p'})\},$$

subject to:

$$c + \tau + \frac{M'}{p} = y(\tau) + \frac{M}{p}.$$

- Using the FOC and envelope yields:

$$\frac{1}{c_t p_t} = \beta \frac{1}{c_{t+1} p_{t+1}} + \gamma \frac{1}{M_{t+1}}. \quad (1)$$

  This may not look meaningful but rewrite as follows:

$$1 = \underbrace{\beta \frac{c_t}{c_{t+1}}}_{1/(1+r)} \underbrace{\frac{p_t}{p_{t+1}}}_{1/(1+\pi)} + \gamma \underbrace{\frac{c_t p_t}{M_{t+1}}}_{V}. \quad (1)$$

  So we have the real interest rate multiplied by the inflation rate plus the velocity of money. Thus,

$$1 = \frac{1}{1+i_t} + \gamma V.$$

  Or,

$$\frac{i}{1+i} = \gamma V.$$

  So we have the marginal cost of holding money equals the marginal benefit.

- Now if we take the forward solution of equation 1, we solve for $1/p_t$:

$$\frac{1}{p_t} = \gamma [y(\tau_t) - g_t] \cdot \underbrace{\sum_{j=0}^{\infty} \beta^j \frac{1}{M_{t+j+1}}}_{Cagan\ Effect}. \quad (2)$$

  The Cagan effect shows that current prices are a function of the expectation of all future quantities of money.

- Now the government's problem.

$$Max_{\{M_{t+1},\tau_t\}} \sum \beta_t u\left(y(\tau_t) - g_t, M_{t+1} \underbrace{[\gamma(y(\tau_t) - g_t) \sum_{j=0}^{\infty} \beta^j M_{t+j+1}^{-1}]}_{1/p_t} \right),$$

  subject to:

$$g_t = \tau_t + \frac{M_{t+1} - M_t}{p_t}.$$

- However, we CANNOT write down the Bellman's equation for this problem because it is NOT recursive. Notice that the time t state variables depend on future values of control variables (M_{t+1}). Thus Bellman's Principal of Optimality does NOT hold. We get a time inconsistency problem here which means the policy rules are not self-inforcing. Given the choice to reoptimize at some later date, the government will choose a new policy.

- Solutions to the time inconsistency problem include contracts (binding the government's hands) and devising arrangements to make the policies self-inforcing.

9.2 Recursive Rational Expectations Equilibria

- We will study two examples: competitive equilibrium and a duopoly.

Competitive Equilibrium - Adjustment Costs of Production

- Suppose there are a large number, n, of firms, all identical.

- Firms choose output, y, to maximize profit.

- The single firm's problem is:

$$Max \sum_{t=0}^{\infty} \beta^t R_t.$$

Where,

$$R_t = p_t y_t \quad \underbrace{-\frac{d}{2}(y_{t+1} - y_t)^2}_{Quadratic\ Adjustment\ Cost} \quad .$$

Assume y_0 is given.

- Now we need to make some conjectures: Conjecture that the competitive price function (demand curve) is linear:

$$p_t = A_0 + A_1 Y_t,$$

with $Y_t = ny_t$ and $A_0, A_1 > 0$. Also conjecture that the law of motion of the aggregate output of the economy is linear:

$$Y_{t+1} = H_0 + H_1 Y_t \equiv H(Y_t),$$

with Y_0 given.

- Given the demand function, the firm's profits are thus:

$$R_t = (A_0 + A_1 Y_t)y_t - \frac{d}{2}(y_{t+1} - y_t)^2.$$

For large n, the firm takes Y_t as given since the firm is atomistic. In equilibrium it will be the case that $Y_t = ny_t$, but at the decision phase, Y_t is taken as given.

- Thus the firm's Dynamic Programming Problem (DPP) is:

$$V(y, Y) = Max_{y'} \ \{(A_0 + A_1 Y)y - \frac{d}{2}(y - y')^2 + \beta V(y', Y')\},$$

  such that:

$$Y' = H(Y).$$

  So y is an endogeneous state and Y is an exogeneous state.

- The FOC yields:

$$\underbrace{d(y' - y)}_{MC} = \underbrace{\beta[A_0 + A_1 H(Y) + d(y'' - y')]}_{MB}.$$

  This means the marginal cost of adjustment equals the marginal benefit. More next time.

10 Lecture 10: March 2, 2005

10.1 Recursive Rational Expectations Equilibrium

Representative Agent Problem - Competitive

- Recall the firm's Dynamic Programming Problem (DPP) is:

$$V(y,Y) = Max_{y'} \ \{(A_0 + A_1 Y)y - \frac{d}{2}(y - y')^2 + \beta V(y', Y')\},$$

  such that:

$$Y' = H(Y).$$

- The FOC yields an Euler:

$$\underbrace{d(y' - y)}_{MC} = \underbrace{\beta[A_0 + A_1 H(Y) + d(y'' - y')]}_{MB}.$$

- So the optimal production plan is a sequence, $\{y_{t+1}\}_0^\infty$ such that:
  - (a) The euler holds taking $H(Y)$, the conjectured aggregate law of motion, and (y_0, Y_0) as given.
  - (b) Terminal Condition: $lim_{t\to\infty} \beta^t y_t(p_t + d(y_{t+1} - y_t)) = 0$. This says the PDV of the benefits of future sales must be zero.
- The solution to this problem will be of the form $y_{t+1} = h(y_t, Y_t)$, so the firm's level of production depends on the lagged state space.
- The ACTUAL law of motion is then:

$$Y_{t+1} = nh(\frac{Y_t}{n}, Y_t) = h(Y_t, Y_t) \text{ if } n = 1.$$

- So the recursive equilibrium is a value function, $V(y,Y)$, a policy function, $h(y,Y)$, and a conjectured law of motion, $H(Y)$, such that:
  - (a) Given $H(Y)$, $V(y,Y)$ solves the firm's DPP with $h(y,Y)$ as the optimal policy function.
  - (b) The law of motion satisfies $H(Y) = h(Y,Y)$ identically in Y.
- So we can think of the actual law of motion as some function of the conjecture where the mapping of the function is the Bellman's equation.

Representative Agent Problem - Social Planner

- Is there a way to rewrite the competitive problem above as a social planners problem such that we get the same optimal policy function? In this case, yes.

- Denote the payoff function of the industry planner (note that we only deal in aggregates now):

$$S(Y,Y') = \int_0^Y A_0 - A_1 x dx - \frac{d}{2}(Y'-Y)^2.$$

So the planner looks at the area under the inverse demand curve between 0 and the level of output, Y. He maximizes the sum of consumer and producer surplus. We still deduct the the adjustment costs of production since those are still present in the aggregate. Thus,

$$S(Y,Y') = A_0 x - \frac{A_1}{2}x^2 \bigg|_0^Y - \frac{d}{2}(Y'-Y)^2.$$

$$S(Y,Y') = A_0 Y - \frac{A_1}{2}Y^2 - \frac{d}{2}(Y'-Y)^2.$$

- Thus the planner's problem is to choose $\{Y_{t+1}\}_0^\infty$ to:

$$Max \sum_{t=0}^{\infty} \beta^t S(Y_t, Y_{t+1}).$$

- The DPP is then:

$$V(Y) = Max_{Y'} \{A_0 Y - \frac{A_1}{2}Y^2 - \frac{d}{2}(Y'-Y)^2 + \beta V(Y')\}.$$

- The FOC yields an euler:

$$d(Y'-Y) = \beta[A_0 - A_1 Y' + d(Y''-Y')],$$

which is precisely the same as the one we found for the individual firm except now we are at the aggregate level. If you think of $n=1$, then you can replace the Y's with y's.

- Note in this case we solved for the H's in the conjectured law of motion directly. We don't have to aggregate up from the individual firm. Thus, if the conjecture does not match the aggregate firm level law of motion, in practice we will use some rule to devise a new conjecture. In the planner's problem we don't have that issue since we are always working at the aggregate level.

Recursive Competitive Equilibrium - General Formulation

- In this setting, remember that the representative agent acts atomisticly. Denote:

$$\begin{aligned} x &\equiv \text{vector of state variables under control of the representative agent} \\ X &\equiv \text{vector of aggregate state variables outside the control of the representative agent} \\ Z &\equiv \text{vector of exogeneous states} \\ u &\equiv \text{vector of agent's controls} \end{aligned}$$

- The agent's DPP:

$$V(x, X, Z) = Max_u \{R(x, X, Z, u) + \beta V(x', X', Z')\},$$

subject to:

$$x' = g(x, X, Z, u) \ \Rightarrow \text{agent's constraints.}$$
$$X' = G(X, Z) \ \Rightarrow \text{conjectured agg. laws of motion.}$$
$$Z' = J(Z) \ \Rightarrow \text{exogeneous states laws of motion.}$$

- Solution to this problem is of the form:

$$u = h(x, X, Z),$$

the individual agent's policy function.

- Using $h(\cdot)$, the ACTUAL law of motion for the aggregate states is then (assuming $n = 1$):

$$X' = G_A(X, Z) = g(X, X, Z, \underbrace{h(X, X, Z)}_{u}).$$

- **Definition:** Thus a recursive rational expectations equilibrium is a value function, $V(\cdot)$, a policy function, $h(\cdot)$, a conjectured law of motion, $G(\cdot)$, and an actual law of motion $G_A(\cdot)$, such that:
  - (a) Given $G(\cdot)$, $V(\cdot)$ solves the firm's DPP with $h(\cdot)$ as the optimal policy function.
  - (b) By evaluating $h(\cdot)$ at $x = X$, $G_A(\cdot) = G(\cdot)$ identically in X.
- Note that we cannot always write the competitive problem as a social planner's problem especially with heterogeneous agents.

10.2 Markov Perfect Equilibrium

Dynamic Duopoly Example

- Consider two firms with payoff function:

$$R_{i,t} = p_t y_{i,t} - \frac{d}{2}(y_{i,t+1} - y_{i,t})^2, \ i = 1, 2.$$

- Total market demand is:

$$p_t = A_0 + A_1(y_{1,t} + y_{2,t}).$$

- Key point: Assume each firm makes a conjecture of the policy function of the other firm. So,

$$y_{-i,t+1} = f_{-i}(y_{i,t}, y_{-i,t}).$$

- Thus the Bellman's equation for firm i is:

$$V_i(y_{i,t}, y_{-i,t}) = Max_{y_{i,t+1}} \{R_{i,t} + \beta V_i(y_{i,t+1}, y_{-i,t+1})\},$$

  subject to:

$$y_{-i,t+1} = f_{-i}(y_{i,t}, y_{-i,t}).$$

- So more on this next time but we have a set up which is time consistent because the firm's choose their quantities simultaneously, so we have something like a cournot Nash equilibrium. Also, since the state space includes only last period's levels of output, the equilibrium will be a Markov equilibrium since the entire history doesn't matter.

11 Lecture 11: March 4, 2005

11.1 Markov Perfect Equilibrium

- Recall our DPP from last time:

$$V_i(y_{i,t}, y_{-i,t}) = Max_{y_{i,t+1}} \{R_{i,t} + \beta V_i(y_{i,t+1}, y_{-i,t+1})\},$$

  subject to:

$$y_{-i,t+1} = f_{-i}(y_{i,t}, y_{-i,t}).$$

- **Definition:** A Markov perfect equilibrium is a pair of value functions, $[V_i(\cdot), V_{-i}(\cdot)]$, and a pair of policy functions, $[f_i(\cdot), f_{-i}(\cdot)]$, such that, given $f_{-i}(\cdot)$, the pair, $[V_i(\cdot), f_i(\cdot)]$, solves agent i's Bellman equation for $i = 1, 2$. Note we take into account the conjecture of the other's policy function, not the actual demand or supply they end up with. Players act simultaneously so there is no time inconsistency.

- There are two methods for solving a monster like this. You could either take a conjecture and iterate on your own Bellman until convergence. Find your own optimal policy. Use this as the conjecture for the other player. Iterate on his Bellman until convergence and see if the policy function that comes out matches your original conjecture. If not, use the other player's policy function as your new conjecture. Repeat until conjectures match the outcomes. The other method involves solving the Bellman's simultaneously, taking the policies that come out of each player's iteration to be the conjecture of the other player's Bellman in the next iteration. This method is more efficient than the first.

11.2 Arrow-Debreu Economy (Complete Market of Contingent Claims)

- In this setup we have stochastic, non-storable, income and we want to devise a market structure such that agents insure each other so there is no uncertainty and consumption can be perfectly smoothed.

- What sort of institutions do we need? – Enforcement. If we are going to have these insurance contracts floating around, we need to make sure they are enforceable when the economy goes sour. Credit markets, in general, are fragile things. You pay someone for a promise that they will pay you back something else at a set date. You don't really take on any collateral in the agreement. The mechanism of enforcement is assumed here but it is a fairly strong assumption. We will also need prices for the contingent claims.

- In the original A-D economy, the market is only open for one day. All trades take place and then everything basically proceeds deterministically from then on. There is also a recursive form of the problem called the "Arrow's Securities Economy" which we will come to later. In this setup, the market is open everyday and we sell one period contingent claims. Prices of the securities will turn out to be the same in both economies, but in the later, we can write down a Bellman which is always cool to do.

Economic Environment - Preferences and Endowments

- Assume we have a stochastic structure to our economy which follows a Markov chain with transition probabilities: $\pi(s'|s)$. This markov process induces probabilities of "histories" of events. Denote:

$$\pi(s_t) \equiv \text{Probability of realizing some state } s \text{ at time } t.$$

$$\pi(s^t) \equiv \text{Probability of the entire history of events from } s_0 \text{ to } s_t.$$

Thus, the unconditional probability of the history s^t is:

$$\pi(s^t) = \pi(s_t, s_{t-1}, \ldots, s_1, s_0) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2}) \cdots \pi(s_1|s_0)\pi(s_0).$$

And the conditional probability of the s^t history given we start at s_0 would be:

$$\pi(s^t|s_0) = \frac{\pi(s^t)}{\pi(s_0)}.$$

Finally, if we know one history has occured and we want to know the probability of seeing future states beyond the current history, we need only look at the conditioning upon the first history (markov property):

$$\pi(s^t|s^\tau) = \pi(s_t|s_{t-1}) \cdots \pi(s_{\tau+1}|s_\tau).$$

- Assume there are I agents in our economy with endowments:

$$y_t^i = y^i(s_t).$$

This assumption that income is a function of the current state and NOT the entire history of states will be crucial later. In fact the whole point of this economy relies on this fact. Assume that every agent observes everyone else's income draws. Perfect information across the agents.

- Agents buy a stream of claims:

$$c^i = \{c_t^i(s^t)\}_0^\infty,$$

to maximize their expected utility. Notice that consumption is potentially a function of the entire history of states. Thus the agent's problem is:

$$U(c^i) = Max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t))\pi(s^t|s_0).$$

So we sum over all time and all histories. Assume $u(\cdot)$ satisfies all the usual conditions as well as the inada conditions.

- The price of the claims traded on day zero is denoted as:

$$q_t^0(s^t),$$

where the superscript is the date of trade (always zero in this case) and the subscript is the delivery date. Of course the price will be a function of the history s^t.

- Thus the agent's budget constraint is:

$$\underbrace{\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^i(s^t)}_{Lifetime\ Spending} \leq \underbrace{\overbrace{\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y^i(s_t)}^{Wealth}}_{Expected\ Lifetime\ Income}.$$

- The key to all of this is that income depends only on a realization of a state and consumption potentially depends on an entire history. Note that we have a SINGLE constraint here on the agent's wealth. In the Arrow's Securities model, we will have a period by period budget constraint.

Economic Environment - Equilibrium

- Consider the agent's problem:

$$U(c^i) = Max \sum_{t=0}^{\infty}\sum_{s^t} \beta^t u(c_t^i(s^t))\pi(s^t|s_0),$$

subject to:

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^i(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y^i(s_t).$$

- Lagrangian for consumer i:

$$\mathcal{L} = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t u(c_t^i(s^t))\pi(s^t|s_0) + \mu^i\left[\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)[y^i(s_t) - c_t^i(s^t)]\right].$$

- FOC (c_t^i):

$$\beta^t u'(c_t^i(s^t))\pi(s^t|s_0) = \mu^i q_t^0(s^t).$$

If we consider the overall utility of this consumer, we can also write the FOC as:

$$\frac{\partial U(c^i)}{\partial c_t^i(s^t)} = \mu^i q_t^0(s^t).$$

This form is a bit more instructive. The marginal "total utility" of consumer i at date t given history s^t is equal to the price of claims times his marginal utility of wealth. This is an intuitive result.

- **Definition:** A competitive equilibrium in the A-D economy is a sequence of consumption, $\{c_t^i(s^t)\}_0^\infty$, for all agents and a sequence of claim prices, $\{q_t^0(s^t)\}_0^\infty$, such that:
  - (1) Markets clear: $\sum_{i=1}^I y^i(s_t) \geq \sum_{i=1}^I c_t^i(s^t)$.
  - (2) Given prices, $\{q_t^0(s^t)\}$, the sequence, $\{c_t^i(s^t)\}$, solves the agent's maximization problem for all i.

Three Properties of the Competitive Equilibrium

- **Property 1** Ratios of multipliers (marginal utilities of wealth), $\frac{\mu^i}{\mu^j}$, are constant across all TIMES and STATES for all i, j. Thus,

$$\frac{\mu^i}{\mu^j} = \frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} \equiv \text{constant}.$$

- **Property 2** Equilibrium consumption allocations are history INDEPENDENT. Rearranging the result in property 1:

$$u'(c_t^i(s^t)) = \frac{\mu^i}{\mu^j} u'(c_t^j(s^t)).$$

$$c_t^i(s^t) = u'^{-1}\left[\frac{\mu^i}{\mu^j} u'(c_t^j(s^t))\right].$$

Since this is true for all i, j, let $j = 1$,

$$c_t^i(s^t) = u'^{-1}\left[\frac{\mu^i}{\mu^1} u'(c_t^1(s^t))\right].$$

Sum this over all consumers and plug it into the resource constraint:

$$\sum_{i=1}^I u'^{-1}\left[\frac{\mu^i}{\mu^1} u'(c_t^1(s^t))\right] \leq \sum_{i=1}^I y^i(s_t).$$

Since income on the RHS is history independent, it means that at equilibrium, consumption must also be history independent:

$$c_t^i(s^t) = c^i(s_t).$$

- For simplicity, since the units of the price system are arbitrary, one of the multipliers can be normalized. So let:

$$\mu^1 = u'(c^1(s_0)).$$

So from our FOC:

$$\beta^t u'(c^1(s_t))\pi(s^t|s_0) = \mu^1 q_t^0(s^t).$$

At $t = 0$:

$$\underbrace{\beta^0}_{1}\underbrace{u'(c^1(s_0))}_{\mu^1}\underbrace{\pi(s^0|s_0)}_{1} = \mu^1 q_0^0(s^0).$$

$$q_0^0(s_0) = 1.$$

But then from property 1:

$$\frac{\mu^i}{\mu^j} = \frac{u'(c^i(s^t))}{u'(c^j(s_t))}.$$

$$\frac{\mu^i}{\mu^1} = \frac{u'(c^i(s^t))}{u'(c^1(s_t))}.$$

At $t = 0$:

$$\frac{\mu^i}{\mu^1} = \frac{u'(c^i(s_0))}{u'(c^1(s_0))}.$$

$$\mu^i = u'(c^i(s_0)).$$

- We will return to property 3 later, but first some examples.

Example 1: Complete Risk Pooling

- Consider a utility function:

$$u(c) = \frac{1}{1-\gamma}c^{1-\gamma}.$$

Property 1 implies:

$$\frac{\mu^i}{\mu^j} = \frac{u'(c^i(s^t))}{u'(c^j(s_t))} = \left(\frac{c_t^i}{c_t^j}\right)^{-\gamma}.$$

So,

$$c_t^i = c_t^j\left(\frac{\mu^i}{\mu^j}\right)^{-1/\gamma}.$$

So we have the consumption of consumer i at time t is a function of the ratio of marginal utilities of wealth.

- So, if we considered two countries, we should see that that the correlation between consumption is perfect. In complete markets, people may have more wealth than others, but changes in wealth should move together. In reality, income is MORE correlated than consumption across countries. A hypothesis of risk-sharing is rejected along with the A-D economy!

Example 2: No Aggregate Uncertainty

- Assume s_t is markov with $s_t \in [0,1]$. Two agents receive income:

$$y_t^1 = s_t, \;\; y_t^2 = 1 - s_t.$$

- We should see that consumers smooth consumption completely. Note that $\sum_i y^i(s_t) = 1$ in all states. There are not situations where both agent's get a bad draw.

- Let's solve this via guess and verify. Conjecture that $c_t^i = c_0^i$ for all t and for $i = 1, 2$. So we guess perfect consumption smoothing.

- Solve our FOC for the price of the claim:

$$q_t^0(s^t) = \frac{\beta^t u'(c_0^i)\pi(s^t|s_0)}{\mu_i}, \;\forall\, t, \; i = 1, 2.$$

If these prices clear the market, we are golden.

- Consider agent i's budget constraint:

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^i(s^t) = \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y^i(s_t).$$

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)[c_t^i(s^t) - y^i(s_t)] = 0.$$

$$\sum_{t=0}^{\infty}\sum_{s^t} \frac{\beta^t u'(c_0^i)\pi(s^t|s_0)}{\mu_i}[c_t^i(s^t) - y^i(s_t)] = 0.$$

$$\underbrace{\frac{u'(c_0^i)}{\mu_i}}_{=1 \; by \; properties} \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi(s^t|s_0)[c_t^i(s^t) - y^i(s_t)] = 0.$$

$$\sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi(s^t|s_0)c_0^i = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi(s^t|s_0)y^i(s_t).$$

So here we have substituted in our conjectured consumption which is time and state independent. Thus the sum on the LHS is just the infinite sum of a constant. So:

$$\sum_{t=0}^{\infty} \beta^t c_0^i = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi(s^t|s_0)y^i(s_t).$$

$$\frac{c_0^i}{1-\beta} = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi(s^t|s_0)y^i(s_t).$$

$$c_0^i = (1-\beta)\underbrace{\sum_{t=0}^{\infty}\sum_{s^t}\beta^t \pi(s^t|s_0) y^i(s_t)}_{Wealth}.$$

So in this setup, we have that each agent consumes a constant fraction, $1-\beta$, of their wealth.

- Now substitute our optimal consumption into the resource constraint:

$$c_0^1 + c_0^2 = (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t}\beta^t \pi(s^t|s_0)[\underbrace{y^1(s_t) + y^2(s_t)}_{1}].$$

$$c_0^1 + c_0^2 = (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t}\beta^t \pi(s^t|s_0).$$

$$c_0^1 + c_0^2 = (1-\beta)\sum_{t=0}^{\infty}\beta^t.$$

$$c_0^1 + c_0^2 = 1.$$

Markets clear!

- So our guess is verified. Agent's consume a constant fraction of their wealth every period (though this fraction may be different between consumers). No aggregate uncertainty. In reality, this means that ratios of consumption should match ratios of wealth between individuals. Of course, as with all things "macro", this fails to show up in actual analysis.

12 Lecture 12: March 4, 2005

12.1 Arrow Debreu and the Theory of Asset Pricing

- Now are still working in the A-D environment, but we will transform our setup into an asset pricing environment.
- Consider the pricing of redundant assets. What does this mean? We bundle together A-D securities (contingent claims) such that they are designed to yield a particular stream of payments:
$$\{d(s_t)\}_0^\infty.$$
Think of these as dividends as a funtion of the current state.
- Define an arbitrage condition:
$$a_0^0 = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t) d(s_t).$$
Where a is the price of an asset priced at time 0 (superscript) whose payoff starts at time 0 (subscript). This price should equal the PDV of our dividend stream of A-D securities priced at the A-D price, q. Note because our price, q_t^0, is dated at time 0 (date of trade - superscript), discounting is already included!
- Consider 4 examples of assets which we can tranform into A-D securities:
  - (1) Riskless Consol: this asset pays 1 unit every period, forever:
  $$d_t(s_t) = 1 \ \forall \ t, s_t.$$
  Thus,
  $$a_0^0 = \sum_{t=0}^{\infty} \sum_{s^t} q_t^0(s^t).$$
  - (2) Riskless Strips: this asset pays off only at one date:
  $$d_\tau = \begin{cases} 1 & if \ \tau = t \\ 0 & else \end{cases}$$
  Thus,
  $$a_0^0 = \sum_{s^t} q_t^0(s^t).$$
  - (3) Tail Asset: this asset pays a dividend stream but only starting after some future date:
  $$\{d_t(s_t)\}_{t \geq \tau}.$$

Thus,

$$a_\tau^0(s^\tau) = \sum_{t=\tau}^{\infty} \sum_{\tilde{s}^t : \tilde{s}^\tau = s^\tau} q_t^0(\tilde{s}^t) d(\tilde{s}_t).$$

So we have the date zero price of an asset that starts payoff at date τ given history s^τ.

- Consider the tail asset. How could we take this price and renormalize it into a date τ price? We just have to divide by the A-D price at date τ: Thus,

$$a_\tau^\tau = \frac{a_\tau^0(s^\tau)}{q_\tau^0(s^\tau)} = \sum_{t=\tau}^{\infty} \sum_{\tilde{s}^t : \tilde{s}^\tau = s^\tau} \frac{q_t^0(\tilde{s}^t)}{q_\tau^0(s^\tau)} d(\tilde{s}_t).$$

But this ratio of A-D prices can also be written:

$$\frac{q_t^0(\tilde{s}^t)}{q_\tau^0(s^\tau)} = q_t^\tau(s^t) = \frac{\beta^t u'(c_t^i(s^t))\pi(s^t|s_0)}{\beta^\tau u'(c_\tau^i(s^\tau))\pi(s^\tau|s_0)} = \beta^{t-\tau} \frac{u'(c_t^i(s^t))}{u'(c_\tau^i(s^\tau))} \pi(s^t|s^\tau).$$

Note:

$$\frac{\pi(s^t|s_0)}{\pi(s^\tau|s_0)} = \frac{\pi(s^t)/\pi(s_0)}{\pi(s^\tau)/\pi(s_0)} = \frac{\pi(s^t)}{\pi(s^\tau)} = \pi(s^t|s^\tau).$$

- Substituting this into the price of our security:

$$a_\tau^\tau = \sum_{t=\tau}^{\infty} \sum_{\tilde{s}^t : \tilde{s}^\tau = s^\tau} q_t^\tau(s^t) d(\tilde{s}_t).$$

- **Property 3** From the ratio of A-D prices above, we have our third property of the A-D equilibrium: Equilibrium prices at time $t \geq 0$, given history s^t, expressed in date τ prices where $0 \leq \tau \leq t$, with history s^τ, are NOT history dependent, ie:

$$q_t^\tau(s^t) = q_k^j(\tilde{s}^k) \text{ for } j, k \geq 0,$$

IF:

  - (1) $t - \tau = k - j$.
  - (2) $[s_t, s_{t-1}, \ldots, s_\tau] = [\tilde{s}_k, \tilde{s}_{k-1}, \ldots, \tilde{s}_j]$.

This property is really why consumption paths turn out to be history independent.

- The last asset that we want to look at and try to translate it into an A-D security is a regular equity. In this case, d will be a stream of dividends paid by the firm. Plug this into the arbitrage condition and this gives the asset pricing condition (asset pricing kernel). So our 4^{th} example:

– (4) One-Period Equity Returns. Using the result above, for an asset that is priced at date τ and pays off in period $\tau + 1$,

$$q^{\tau}_{\tau+1}(s^{\tau+1}) = \beta \frac{u'(c^i_{\tau+1})}{u'(c^i_{\tau})} \pi(s_{\tau+1}|s_{\tau}).$$

Assume the asset pays a random amount $W(s_{\tau+1})$ so the asset's price becomes:

$$\begin{aligned} p^{\tau}_{\tau}(s^{\tau}) &= \sum_{s^{\tau+1}} q^{\tau}_{\tau+1}(s^{\tau+1}) W(s_{\tau+1}) \\ &= \sum_{s^{\tau+1}} \beta \frac{u'(c^i_{\tau+1})}{u'(c^i_{\tau})} \pi(s_{\tau+1}|s_{\tau}) W(s_{\tau+1}) \\ &= E_{\tau}\left[\beta \frac{u'(c^i_{\tau+1})}{u'(c^i_{\tau})} W(s_{\tau+1})\right] \end{aligned}$$

- We set things up like this because we will need these different returns when we move to the recursive setup of A-D.

12.2 Arrow Securities Economy

- This is just a recursive form of the A-D economy. Denote:

$$\theta^i_t \equiv \text{Claims on date } t \text{ consumption goods that agent } i \text{ brings to period } t.$$

$$Q(s_{t+1}|s_t) \equiv \text{Price of 1 unit of } c_{t+1} \text{ at } t+1 \text{ contingent on } s_{t+1} \text{ given } s_t.$$

- Thus agent i's, date t, budget constraint is:

$$c^i_t(s^t) + \sum_{s_{t+1}} Q(s_{t+1}|s_t)\theta^i_{t+1} \leq y^i(s_t) + \theta^i_t(s_t).$$

  Note this is NOT a wealth constraint, but rather a period by period constraint. We have income and claims brought forward on the RHS and consumption and holdings of claims until next period on the LHS. The agent chooses consumption, $c^i_t(s^t)$ and portfolios of 1 period assets, $\{Q^i_{t+1}(s_{t+1})\}$.

- Next time we will set up the Bellman's equation for this problem. Hopefully it will be more clear then.

13 Lecture 13: March 7, 2005

13.1 Arrow's Securities

- A note on the A-D economy: perfect consumption smoothing comes from consumers being able to perfectly insure away all individual income uncertainty. If there are aggregate shocks, we may not get perfect smoothing.

- Recall our budget constraint from last time:

$$c_t^i(s^t) + \sum_{s_{t+1}} Q(s_{t+1}|s_t)\theta_{t+1}^i(s_{t+1}) \le y_t^i(s_t) + \theta_t^i(s_t).$$

  Think of the "lucas apple tree" idea. y_t is the number of apples awarded in period t, θ_t is the number of apples the agent had claims to. With that amount of wealth, the agent can either consume the apples, c_t, or else use them to buy contracts on future apples (valued at Q)! Note that θ^i can be positive or negative.

- We need to rule out Ponzi games in this situation. This is also called imposing a "Natural Debt Limit". Since the inada conditions hold, consumers have infinite marginal utility when consumption is close to zero. Hence they will avoid the possibility of this situation at ALL costs. Thus the natural debt limit should be the upper bound on what they can promise to pay in any state of nature, s_{t+1}.

- So consider a "Tail" asset of agent i's income:

$$A_t^i(s^t) = \sum_{t=\tau}^{\infty} \sum_{\tilde{s}^\tau : \tilde{s}^t = s^\tau} q_\tau^t(\tilde{s}^\tau) y^i(\tilde{s}_\tau).$$

  Note this is exactly the formula we had in previous lectures with $d(\cdot) = y(\cdot)$. Thus the value of the asset equals the PDV of the stream of agent i's income. Note that both the income of the agent and the A-D price are history independent (Property 3). Thus, at equilibrium, the value of this asset will be time independent, ie:

$$A_t^i(s^t) = \overline{A}^i(s_t).$$

  So $-\overline{A}^i(s_t)$ will be our natural debt limit because agents would never want their assets to fall below this as they could face infinite marginal utility.

- Recall, we can rewrite the A-D price as:

$$q_\tau^t(s^\tau) = \frac{q_\tau^0(s^\tau)}{q_t^0(s^t)} = \beta^{\tau - t} \frac{u'(c_\tau^i(s^\tau))}{u'(c_t^i(s^t))} \frac{\pi(s^\tau)}{\pi(s^t)}.$$

  Substitute this into our pricing equation above:

$$A^i_t(s^t) = \sum_{t=\tau}^{\infty} \sum_{\tilde{s}^\tau:\tilde{s}^t=s^\tau} \beta^{\tau-t} \frac{u'(c^i_\tau(s^\tau))}{u'(c^i_t(s^t))} \frac{\pi(s^\tau)}{\pi(s^t)} y^i(\tilde{s}_\tau).$$

- At equilibrium, this becomes:

$$\overline{A}^i(s) = \sum_{t=\tau}^{\infty} \sum_{\tilde{s}^\tau:\tilde{s}^t=s^\tau} \beta^{\tau-t} \frac{u'(\overline{c}(s'))}{u'(\overline{c}(s))} \frac{\pi(s^\tau)}{\pi(s^t)} y^i(\tilde{s}_\tau).$$

  Which can be written in dynamic form:

$$\overline{A}^i(s) = y^i(s) + \beta E_s \left[\frac{u'(\overline{c}(s'))}{u'(\overline{c}(s))} \overline{A}^i(s') \right].$$

- So, we can now write out the individual household's dynamic programming problem (DPP).

$$\text{States: } \{\theta^i_t, y^i(s_t)\}.$$

$$\text{Policy Functions: } \{c^i = h^i(\theta^i, s),\ \ \theta^i_{t+1}(s_{t+1}) = g^i(\theta^i, s, s')\}.$$

  Bellman's Equation:

$$V^i(\theta, s) = Max \left\{ u(c) + \beta \sum_{s'} \pi(s'|s) V^i(\theta'(s'), s') \right\},$$

  subject to:

$$(1)\ \ c + \sum_{s'} \theta(s') Q(s'|s) \le y^i(s) + \theta,$$

$$(2)\ \ \theta(s') \ge -\overline{A}^i(s'),$$

$$(3)\ \ c \ge 0.$$

  Where the second condition is our "no ponzi" game constraint.

- **Definition:** A Recursive Competitive Equilibrium in the Arrow Securities Economy. Denote the distribution of agent's wealth as:

$$\vec{\theta}_t = \{\theta^i_t\}_{i=1}^I, \text{ with: } \sum_{i=1}^I \theta^i_t = 0.$$

  Since we have a closed economy, one person's assets are another's liabilities. An equilibrium is given by an initial distribution of wealth, $\vec{\theta}_0$, a pricing kernel for one period claims, $Q(s'|s)$, and a set of value functions, $V^i(\cdot)$, with associated policy functions, $(h^i(\cdot), g^i(\cdot))$, for each agent such that:

  - (1) For all agents, given their initial wealth and prices, the agent's policy functions solve their DPP.

- (2) For all realizations, $\{s_t\}_0^\infty$, the allocations,

$$\left\{\left\{c_t^i, \{\theta_{t+1}^i(s')\}_{s'}\right\}_i\right\}_t,$$

and the policy functions in (1) satisfy market clearing in both the goods and asset markets:

$$\sum_i c_t^i = \sum_i y^i(s_t), \text{ and } \sum_i \theta_{t+1}^i(s') = 0 \; \forall \; t, s'.$$

- The FOC and envelope conditions of the DPP above yield an euler:

$$\underbrace{u'(c_t^i)Q(s_{t+1}|s_t)}_{MC\ of\ Purchasing\ \theta} = \underbrace{\beta u'(c_{t+1}^i)\pi(s_{t+1}|s_t)}_{MB}.$$

13.2 Equivalence of the Arrow Debreu and Arrow Securities Economies

- We would like to show that the allocations in the A-D and A-S economies are equivalent. We'll do this in a sequence of many, many, complicated steps ....
  - *Step 1* Take the prices from the A-D economy, $q_t^0(s^t)$, and conjecture:

    $$Q(s_{t+1}|s_t) = q_{t+1}^t(s^{t+1}) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)}.$$

    So we are guessing that the prices are the same for one period claims.
  - *Step 2* Take the consumption allocations from the A-D economy:

    $$\{c_t^i(s^t)\}.$$

  - *Step 3* Consider the implications of steps 1 and 2 for the optimality condition of the A-S economy. Substituting:

    $$u'(c_t^i)Q(s_{t+1}|s_t) = \beta u'(c_{t+1}^i)\pi(s_{t+1}|s_t).$$

    $$Q(s_{t+1}|s_t) = \beta\frac{u'(c_{t+1}^i)}{u'(c_t^i)}\pi(s_{t+1}|s_t).$$

    $$\frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = \beta\frac{u'(c_{t+1}^i)}{u'(c_t^i)}\pi(s_{t+1}|s_t).$$

  - *Step 4* Postulate an initial distribution of wealth, $\vec{\theta}_0 = 0$ and show that it supports recursive trading of Arrow Securities for the consumption plans of the Arrow-Debreu economy. Since consumers in the A-D economy initially have NO wealth, we must start the A-S economy with no wealth as well.
  - *Steps 5 - ?* Next time...

14 Lecture 14: March 14, 2005

14.1 Equivalence of A-D and A-S Equilibria

- Recall from last time, we started our proof which took the prices and consumption paths from the A-D economy and plugged them into the optimality condition for the A-S economy. In step 4, we postulated that the initial distribution of wealth was $\vec{\theta}_0 = 0$. Now we continue with step 5:
  - *Step 5* Show $\vec{\theta}_0$ allows agents to afford $\{c_t^i(s^t)\}$, the A-D consumption plans, which means it satisfies the budget constraint with equality. Thus we divide step 5 into several parts.
  - *Step 5a* Define an agent's implied financial wealth. At time $\tau \geq t$, the net claim of agent i can be written:

    $$c_\tau^i(\tilde{s}^\tau) - y^i(\tilde{s}_\tau),$$

    ie, the difference between what he will consume and what he will earn in that period. Thus, the implied finanical wealth is the PDV of this stream of claims from any time period out to infinity. Or:

    $$\Upsilon_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{\{\tilde{s}^\tau : \tilde{s}^t = s^t\}} q_\tau^t(\tilde{s}^\tau)[c_\tau^i(\tilde{s}^\tau) - y^i(\tilde{s}_\tau)].$$

    We know that along an equilibrium, both the A-D prices and the consumption paths will be history independent. Thus we can write:

    $$\Upsilon_t^i(s^t) = \overline{\Upsilon}^i(s_t).$$

    We also know that at any period, the net claims must be zero so,

    $$\sum_{i=1}^{I} \overline{\Upsilon}^i(s_t) = 0.$$

  - *Step 5b* Conjecture that an agent's optimal porfolio choice at time $t+1$ will be:

    $$\theta_{t+1}^i(s_{t+1}) = \overline{\Upsilon}^i(s_{t+1}).$$

– *Step 5c* Thus the value of the agent's portfolio can be written:

$$\begin{aligned}\sum_{s_{t+1}} \theta^i_{t+1}(s_{t+1})Q(s_{t+1}|s_t) &= \sum_{s_{t+1}} \overline{\Upsilon}^i(s_{t+1})Q(s_{t+1}|s_t) \\ &= \sum_{s_{t+1}} \overline{\Upsilon}^i(s_{t+1})q^t_{t+1}(s^{t+1}) \\ &= \sum_{s_{t+1}} q^t_{t+1}(s^{t+1}) \sum_{\tau=t}^{\infty} \sum_{\{\tilde{s}^\tau:\tilde{s}^t=s^t\}} q^{t+1}_\tau(\tilde{s}^\tau)[c^i_\tau(\tilde{s}^\tau) - y^i(\tilde{s}_\tau)] \\ &= \sum_{\tau=t+1}^{\infty} \sum_{\{\tilde{s}^\tau:\tilde{s}^t=s^t\}} q^t_\tau(\tilde{s}^\tau)[c^i_\tau(\tilde{s}^\tau) - y^i(\tilde{s}_\tau)]\end{aligned}$$

Where this last equality come from:

$$q^{t+1}_\tau(s^\tau)q^t_{t+1}(s^{t+1}) = \frac{q^0_\tau(s^\tau)}{q^0_{t+1}(s^{t+1})}\frac{q^0_{t+1}(s^{t+1})}{q^0_t(s^t)} = \frac{q^0_\tau(s^\tau)}{q^0_t(s^t)} = q^t_\tau(s^\tau).$$

– *Step 5d* Use the result from (5c) and the budget constraint of the agent of the A-S economy. Recall the budget constraint:

$$c^i_t(s^t) + \sum_{s_{t+1}} Q(s_{t+1}|s_t)\theta^i_{t+1}(s_{t+1}) = y^i_t(s_t) + \theta^i_t(s_t).$$

At $t = 0$:

$$\hat{c}^i_0(s_0) + \sum_{s_1} \theta_1(s_1)Q(s_1|s_0) = y^i(s_0) + 0.$$

Where we are conjecturing that $\vec{\theta}_0 = 0$, or that the agent starts out with no wealth. Using the result in (5c), we can rewrite this as:

$$\hat{c}^i_0(s_0) + \sum_{t=1}^{\infty}\sum_{s^t} q^0_t(s_t)[c^i_t(s_t) - y^i(s_t)] = y^i(s_0).$$

And at any date and state:

$$\hat{c}^i_t(s_t) + \sum_{s_{t=1}} \theta_{t+1}(s_{t+1})Q(s_{t+1}|s_t) = y^i(s_t) + \overline{\Upsilon}^i(s_t).$$

Using the result in (5c), we can rewrite this as:

$$\hat{c}^i_t(s_t) + \sum_{\tau=t+1}^{\infty} \sum_{\{\tilde{s}^\tau:\tilde{s}^t=s^t\}} q^t_\tau(\tilde{s}_\tau)[c^i_\tau(s_\tau) - y^i(s_\tau)] = y^i(s_t) + \overline{\Upsilon}^i(s_t).$$

Thus,

$$[\hat{c}^i_t(s_t) - y^i(s_t)] + \sum_{\tau=t+1}^{\infty} \sum_{\{\tilde{s}^\tau : \tilde{s}^t = s^t\}} q^t_\tau(\tilde{s}_\tau)[c^i_\tau(s_\tau) - y^i(s_\tau)] = \overline{\Upsilon}^i(s_t).$$

$$\sum_{\tau=t}^{\infty} \sum_{\{\tilde{s}^\tau : \tilde{s}^t = s^t\}} q^t_\tau(\tilde{s}_\tau)[c^i_\tau(s_\tau) - y^i(s_\tau)] = \overline{\Upsilon}^i(s_t).$$

$$\overline{\Upsilon}^i(s_t) = \overline{\Upsilon}^i(s_t).$$

So the budget constraint is satisfied!

- So our consumption plans from the A-D economy can be supported by the optimality condition of the A-S economy with an initial wealth vector of zero.
- Note that because of the implied financial wealth inheriant in all of this, we don't need an additional debt limit because agents will never borrow more than their implied financal lifetime wealth.

14.2 Incomplete Markets

- What if there was only one asset, a treasury bond for example and since agents have uncertain income, they cannot insure away their risk.
- The individual's DPP would be:

$$V(a_h, \overline{s}_i) = Max\ \{u(\underbrace{\overbrace{w\overline{s}_i}^{income} + (1+r)a_h - a'}_{c}) + \beta \underbrace{\sum_{j=1}^{m} P(i,j)}_{m\ poss\ realizations} V(a', \overline{s}_j)\},$$

where,

$$a_h \in \mathcal{A} = \{a_1 < \cdots < a_n\}.$$

Note that w is the agent's wage, r is the interest rate on the asset, s is markov with m possible realizations representing income or employment shocks, and a_h is the agent's position in the asset.

- The solution will be of the form: $a' = g(a_h, \overline{s}_i)$.
- Recall that in the complete market's world, the distribution of wealth for the agents was constant over all time and states. Now the distribution of wealth will be HISTORY DEPENDENT!!
- The economy-wide distribution of wealth, denoted $\lambda(a,s) = Prob(a_t = a, s_t = s)$, is found by looking for an ergodic distribution over the states of the agent's DPP. Market clearing in the debt market implies total borrowing equals total lending, or $\sum_{a,s} \lambda(a,s) g(a,s) = 0$.

Midterm Review

14.3 Key Lecture Notes

- Key quote from Lucas paper - the policies that pursue stabilization have a much smaller impact on welfare than the policies that pursue growth.

- Markov Chains – $\{x_t\}$ satisfies the markov property if $Pr(x_{t+1}|x_t, \ldots, x_{t-k}) = Pr(x_{t+1}|x_t)$. A markov process is characterized by a vector of realizations, a vector of initial probabilities, and a transition matrix. Ergodic distribution: $\pi' = \pi' P$. If P^n for $n \geq 1$ has all strictly positive elements, then there is a single unique ergodic distribution. Expecations ...

$$\text{Unconditional: } E[x_t] = (\pi_0 P^t)\overline{x} = \pi_t'\overline{x}.$$

$$\text{Conditional: } E[x_{t+k}|x_t = \overline{x}] = P^k\overline{x}.$$

- Cagan effect: current prices are a function of the expectation of all future quantities of money.

- Whenever we have time t states depending on future values of the controls (as with a Cagan effect), we cannot write down a bellman's equation because the Bellman principal of optimality fails. Usually we'll get time inconsistency in a situation like this.

- Representitive Agent Problem - Competitive. A recursive equilibrium if a value function and a policy function and a conjectured law of motion such that, given the conjecture, the value function solves the firm's DPP with the policy function as the optimum. The law of motion must satisfy: $H(Y) = h(y = Y, Y)$. Thus the conjecture becomes the actual in equilibrium.

- Markov Perfect Equilibrium - Dynamic Duopoly. Two firms and each has a conjecture of what the other's policy function looks like. So a MPE is a pair of value functions and a pair of policy functions, one for each firm, such that, given each firm's conjecture of the other's policy function, my value and policy function for my firm solves my DPP. Players act simultaneously so there is no time inconsistency problems.

Arrow Debreu Economy

$$\pi(s^t) = \pi(s_t, s_{t-1}, \ldots, s_1, s_0) = \pi(s_t|s_{t-1})\pi(s_{t-1}|s_{t-2}) \cdots \pi(s_1|s_0)\pi(s_0).$$

$$y_t^i = y^i(s_t).$$

$$U(c^i) = Max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c_t^i(s^t))\pi(s^t|s_0).$$

- The price of the claims traded on day zero is denoted as:

$$q_t^0(s^t),$$

- Thus the agent's budget constraint is:

$$\underbrace{\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^i(s^t)}_{Lifetime\ Spending} \leq \underbrace{\overbrace{\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y^i(s_t)}^{Wealth}}_{Expected\ Lifetime\ Income}.$$

- Lagrangian for consumer i:

$$\mathcal{L} = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t u(c_t^i(s^t))\pi(s^t|s_0) + \mu^i \left[\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)[y^i(s_t) - c_t^i(s^t)]\right].$$

- FOC (c_t^i):

$$\beta^t u'(c_t^i(s^t))\pi(s^t|s_0) = \mu^i q_t^0(s^t).$$

- **Definition:** A competitive equilibrium in the A-D economy is a sequence of consumption, $\{c_t^i(s^t)\}_0^{\infty}$, for all agents and a sequence of claim prices, $\{q_t^0(s^t)\}_0^{\infty}$, such that:
  - (1) Markets clear: $\sum_{i=1}^{I} y^i(s_t) \geq \sum_{i=1}^{I} c_t^i(s^t)$.
  - (2) Given prices, $\{q_t^0(s^t)\}$, the sequence, $\{c_t^i(s^t)\}$, solves the agent's maximization problem for all i.

Three Properties of the Competitive Equilibrium

- **Property 1** Ratios of multipliers (marginal utilities of wealth), $\frac{\mu^i}{\mu^j}$, are constant across all TIMES and STATES for all i, j. Thus,

$$\frac{\mu^i}{\mu^j} = \frac{u'(c_t^i(s^t))}{u'(c_t^j(s^t))} \equiv \text{constant}.$$

- **Property 2** Equilibrium consumption allocations are history INDEPENDENT. Rearranging the result in property 1:

$$u'(c_t^i(s^t)) = \frac{\mu^i}{\mu^j} u'(c_t^j(s^t)).$$

$$c_t^i(s^t) = u'^{-1}\left[\frac{\mu^i}{\mu^j} u'(c_t^j(s^t))\right].$$

Since this is true for all i, j, let $j = 1$,

$$c_t^i(s^t) = u'^{-1}\left[\frac{\mu^i}{\mu^1} u'(c_t^1(s^t))\right].$$

Sum this over all I consumers and plug it into the resource constraint:

$$\sum_{i=1}^{I} u'^{-1}\left[\frac{\mu^i}{\mu^1}u'(c_t^1(s^t))\right] \leq \sum_{i=1}^{I} y^i(s_t).$$

$$c_t^i(s^t) = c^i(s_t).$$

- Normalized. So let:

$$\mu^1 = u'(c^1(s_0)).$$

$$q_0^0(s_0) = 1.$$

Example 2: No Aggregate Uncertainty

- Conjecture that $c_t^i = c_0^i$ for all t and for $i = 1, 2$.
- Solve our FOC for the price of the claim:

$$q_t^0(s^t) = \frac{\beta^t u'(c_0^i)\pi(s^t|s_0)}{\mu_i}, \ \forall\, t,\ i = 1, 2.$$

If these prices clear the market, we are golden.

- Consider agent i's budget constraint:

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^i(s^t) = \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y^i(s_t).$$

$$c_0^i = (1-\beta)\underbrace{\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t|s_0)y^i(s_t)}_{Wealth}.$$

$$c_0^1 + c_0^2 = (1-\beta)\sum_{t=0}^{\infty}\sum_{s^t}\beta^t\pi(s^t|s_0)[\underbrace{y^1(s_t) + y^2(s_t)}_{1}] = 1.$$

Arrow Securities Economy

- Define an arbitrage condition:

$$a_0^0 = \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)d(s_t).$$

- (3) Tail Asset:

$$a_\tau^0(s^\tau) = \sum_{t=\tau}^{\infty}\sum_{\tilde{s}^t:\tilde{s}^\tau = s^\tau} q_t^0(\tilde{s}^t)d(\tilde{s}_t).$$

- **Property 3** From the ratio of A-D prices above, we have our third property of the A-D equilibrium: Equilibrium prices at time $t \geq 0$, given history s^t, expressed in date τ prices where $0 \leq \tau \leq t$, with history s^τ, are NOT history dependent, ie:

$$q_t^\tau(s^t) = q_k^j(\tilde{s}^k) \text{ for } j, k \geq 0,$$

IF:

  - (1) $t - \tau = k - j$.
  - (2) $[s_t, s_{t-1}, \ldots, s_\tau] = [\tilde{s}_k, \tilde{s}_{k-1}, \ldots, \tilde{s}_j]$.

This property is really why consumption paths turn out to be history independent.

- Denote:

$$\theta_t^i \equiv \text{Claims on date } t \text{ consumption goods that agent } i \text{ brings to period } t.$$

$$Q(s_{t+1}|s_t) \equiv \text{Price of 1 unit of } c_{t+1} \text{ at } t+1 \text{ contingent on } s_{t+1} \text{ given } s_t.$$

- Thus agent i's, date t, budget constraint is:

$$c_t^i(s^t) + \sum_{s_{t+1}} Q(s_{t+1}|s_t)\theta_{t+1}^i \leq y^i(s_t) + \theta_t^i(s_t).$$

- A note on the A-D economy: perfect consumption smoothing comes from consumers being able to perfectly insure away all individual income uncertainty. If there are aggregate shocks, we may not get perfect smoothing.

- So consider a "Tail" asset of agent i's income:

$$A_t^i(s^t) = \sum_{t=\tau}^{\infty} \sum_{\tilde{s}^\tau : \tilde{s}^t = s^\tau} q_\tau^t(\tilde{s}^\tau) y^i(\tilde{s}_\tau).$$

Note this is exactly the formula we had in previous lectures with $d(\cdot) = y(\cdot)$. Thus the value of the asset equals the PDV of the stream of agent i's income. Note that both the income of the agent and the A-D price are history independent (Property 3). Thus, at equilibrium, the value of this asset will be time independent, ie:

$$A_t^i(s^t) = \overline{A}^i(s_t).$$

So $-\overline{A}^i(s_t)$ will be our natural debt limit because agents would never want their assets to fall below this as they could face infinite marginal utility. Which can be written in dynamic form:

$$\overline{A}^i(s) = y^i(s) + \beta E_s\left[\frac{u'(\overline{c}(s'))}{u'(\overline{c}(s))}\overline{A}^i(s')\right].$$

- So, we can now write out the individual household's dynamic programming problem (DPP).

$$\text{States: } \{\theta_t^i, y^i(s_t)\}.$$

$$\text{Policy Functions: } \{c^i = h^i(\theta^i, s),\ \ \theta_{t+1}^i(s_{t+1}) = g^i(\theta^i, s, s')\}.$$

Bellman's Equation:

$$V^i(\theta, s) = Max \left\{ u(c) + \beta \sum_{s'} \pi(s'|s) V^i(\theta'(s'), s') \right\},$$

subject to:

$$(1)\ \ c + \sum_{s'} \theta(s') Q(s'|s) \leq y^i(s) + \theta,$$

$$(2)\ \ \theta(s') \geq -\overline{A}^i(s'),$$

$$(3)\ \ c \geq 0.$$

- **Definition:** A Recursive Competitive Equilibrium in the Arrow Securities Economy. Denote the distribution of agent's wealth as:

$$\vec{\theta}_t = \{\theta_t^i\}_{i=1}^I, \text{ with: } \sum_{i=1}^I \theta_t^i = 0.$$

Since we have a closed economy, one person's assets are another's liabilities. An equilibrium is given by an initial distribution of wealth, $\vec{\theta}_0$, a pricing kernel for one period claims, $Q(s'|s)$, and a set of value functions, $V^i(\cdot)$, with associated policy functions, $(h^i(\cdot), g^i(\cdot))$, for each agent such that:

  - (1) For all agents, given their initial wealth and prices, the agent's policy functions solve their DPP.
  - (2) For all realizations, $\{s_t\}_0^\infty$, the allocations,

$$\left\{ \left\{ c_t^i, \{\theta_{t+1}^i(s')\}_{s'} \right\}_i \right\}_t,$$

    and the policy functions in (1) satisfy market clearing in both the goods and asset markets:

$$\sum_i c_t^i = \sum_i y^i(s_t), \text{ and } \sum_i \theta_{t+1}^i(s') = 0 \; \forall \; t, s'.$$

Equivalence of the Arrow Debreu and Arrow Securities Economies

  - *Step 1* Take the prices from the A-D economy, $q_t^0(s^t)$, and conjecture:

$$Q(s_{t+1}|s_t) = q_{t+1}^t(s^{t+1}) = \frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)}.$$

– *Step 2* Take the consumption allocations from the A-D economy:

$$\{c_t^i(s^t)\}.$$

– *Step 3* Consider the implications of steps 1 and 2 for the optimality condition of the A-S economy. Substituting:

$$u'(c_t^i)Q(s_{t+1}|s_t) = \beta u'(c_{t+1}^i)\pi(s_{t+1}|s_t).$$

$$Q(s_{t+1}|s_t) = \beta \frac{u'(c_{t+1}^i)}{u'(c_t^i)}\pi(s_{t+1}|s_t).$$

$$\frac{q_{t+1}^0(s^{t+1})}{q_t^0(s^t)} = \beta \frac{u'(c_{t+1}^i)}{u'(c_t^i)}\pi(s_{t+1}|s_t).$$

– *Step 4* Postulate an initial distribution of wealth, $\vec{\theta}_0 = 0$ and show that it supports recursive trading of Arrow Securities for the consumption plans of the Arrow-Debreu economy.

– *Step 5* Show $\vec{\theta}_0$ allows agents to afford $\{c_t^i(s^t)\}$, the A-D consumption plans, which means it satisfies the budget constraint with equality. Thus we divide step 5 into several parts.

– *Step 5a* Define an agent's implied financial wealth.

$$\Upsilon_t^i(s^t) = \sum_{\tau=t}^{\infty} \sum_{\{\tilde{s}^\tau:\tilde{s}^t=s^t\}} q_\tau^t(\tilde{s}^\tau)[c_\tau^i(\tilde{s}^\tau) - y^i(\tilde{s}_\tau)].$$

We know that along an equilibrium, both the A-D prices and the consumption paths will be history independent. Thus we can write:

$$\Upsilon_t^i(s^t) = \overline{\Upsilon}^i(s_t).$$

– *Step 5b* Conjecture that an agent's optimal porfolio choice at time $t+1$ will be:

$$\theta_{t+1}^i(s_{t+1}) = \overline{\Upsilon}^i(s_{t+1}).$$

– *Step 5c* Thus the value of the agent's portfolio can be written:

$$\sum_{s_{t+1}} \theta_{t+1}^i(s_{t+1})Q(s_{t+1}|s_t) = \sum_{\tau=t+1}^{\infty} \sum_{\{\tilde{s}^\tau:\tilde{s}^t=s^t\}} q_\tau^t(\tilde{s}^\tau)[c_\tau^i(\tilde{s}^\tau) - y^i(\tilde{s}_\tau)]$$

– *Step 5d* Use the result from (5c) and the budget constraint of the agent of the A-S economy. Recall the budget constraint:

$$c_t^i(s^t) + \sum_{s_{t+1}} Q(s_{t+1}|s_t)\theta_{t+1}^i(s_{t+1}) = y_t^i(s_t) + \theta_t^i(s_t).$$

And at any date and state:

$$\hat{c}_t^i(s_t) + \sum_{s_{t=1}} \theta_{t+1}(s_{t+1})Q(s_{t+1}|s_t) = y^i(s_t) + \overline{\Upsilon}^i(s_t).$$

Using the result in (5c), we can rewrite this as:

$$\hat{c}_t^i(s_t) + \sum_{\tau=t+1}^{\infty} \sum_{\{\tilde{s}^\tau : \tilde{s}^t = s^t\}} q_\tau^t(\tilde{s}_\tau)[c_\tau^i(s_\tau) - y^i(s_\tau)] = y^i(s_t) + \overline{\Upsilon}^i(s_t).$$

$$\overline{\Upsilon}^i(s_t) = \overline{\Upsilon}^i(s_t).$$

- Recall that in the complete market's world, the distribution of wealth for the agents was constant over all time and states. Now the distribution of wealth will be HISTORY DEPENDENT!!

14.4 Notes from Problem Sets and Exams

- Check for separability of utility functions. Marginal effects may involve other terms (like lagged consumption example).
- Don't make choice varialbe in Bellman equation one that depends on something stochastic (like choosing consumption or next period's asset return when the return is stochastic.
- For CRRA, usually safe to assume $\gamma > 1$.
- For any problem with Q-theory, use lagrangians, not bellmans.
- Marginal Q = Average Q if :
  - (1) The firm is a price taker.
  - (2) The production function is linearly homogeneous in (K, L).
  - (3) The Installation cost function is linearly homogeneous in (K, I).
- No prices in a SP problem!
- In A-D setup, look for no-aggregate uncertainty ... you definitely will get perfect consumption smoothing.
- Consider a natural debt limit in A-S economy.
- For Brock Mirman – optimize over consumption, not K'.
- Normalization in A-D : $\mu^i = u'(c(\lambda_0))$.

Begin Haltiwanger Lectures

15 Lecture 15: March 28, 2005

15.1 Course Outline

- Part 1: Equilibrium Business Cycle (BC) Models. We assume things like perfect competition, markets clearing and optimizing agents. Models will include the Lucas Islands model and a classic BC model. The important thing are the shocks. Are they real or nominal? Persistent?

- Part 2: New Keynesian Macro Models. Introduce coordination problems and externalities. What happens if markets do not clear all the time (wage and price rigidities).

- Part 3: Unemployment. Why doesn't the labor market always clear?

- General theme in BC models: Persistence. See G-15.1. Is output serially correlated? We study not only the shock but also the propagation of the shock over time. For example, recessions tend to be deep and recovery is long. Why is this so?

- We distinguish between the "Lucas Revolution" where all models were solved out completely so we could say exactly what agents did. The common quote is "It takes a model to beat a model." Don't just tell me I'm wrong. Give me something else that works better. Then there are the "New Keynesians" which focus on the idea that market do not always work correctly. There may be multiplier and persistence effects.

- Shocks. There are two possible types of shocks: nominal and real. Some examples:
  - Nominal: Monetary policy changes - this should be neutral but it clearly is not in the data.
  - Real: Technology; Cost - raw materials like oil; Fiscal policy; Changes in market structure or institutions - a foreign economy starts producing a competing product; Confidence shocks - firm's or household's perceptions of the current economic environment.

  In general, even if we know what the shock is, we may not know if the effects will be large or small, and we could have multiple shocks moving things in opposite directions.

- Propagation. Some shocks shouldn't be important but they turn out to be (Sun-Spot Models). If people believe, the shock could be self-fulfilling. So this is why the idea of propagation is important in BC models.

15.2 Lucas Island Models (1975)

- This model considers two issues: 1) why do nominal shocks matter?; 2) the way that agents form expectations are crucial.

- Consider the following "micro-level" supply curve for market i:

$$y_t^i = b(p_t^i - E[p_t|I_t^i]),$$

where y_t^i and p_t^i is market i's output and price. p_t is the general price level which is only estimated using information I_t^i. All lower-case terms are in logs, ie $y_t = log(Y_t)$.

- Agents on island i (or in market i), observe their own price. We could write:

$$p_t^i = p_t + z_t^i,$$

where z_t^i is the idiosyncratic price variation in market i. Since they know p_t^i, they can try to extract p_t using the distributions of p_t and z_t^i. Assume:

$$z_t^i \sim N(0, \sigma_z^2),$$

$$p_t \sim N(E(p_t|I_t), \sigma_p^2),$$

where I_t is common information to all markets (islands). Thus I_t^i contains I_t and p_t^i. Agents know the model, their own price and the distributions. Since they are trying to determine the overall market price, this is called a "Signal Extraction Problem."

- We will show next time:

$$E[p_t|I_t^i] = (1-\theta)E[p_t|I_t] + \theta p_t^i,$$

with,

$$\theta = \frac{\sigma_p^2}{\sigma_z^2 + \sigma_p^2}.$$

So agent's form their expectation of the general price as a weighted average of the common prior and their updated information (their own price).

- Note if the variation in z is large, $\theta \to 0$ so price expections should be based on the common prior only. If the variation in z is small, $\theta \to 1$, and agents should only use their updated information. In general, if σ_z^2 is small relative to σ_p^2, the signal extraction problem is small.

- Note we will be interested in relative prices. Suppose there is a positive monetary shock which increases the general price level along with market i's price. Do the relative prices change? This depends crucially on θ.

- Aggregating the micro supply curve and substituting in from the solution above:

$$\begin{aligned} y_t^i &= b(p_t^i - [\theta p_t^i + (1-\theta)E[p_t|I_t]]) \\ &= b(1-\theta)(p_t^i - E[p_t|I_t]) \\ y_t &= \beta(p_t - E[p_t|I_t]) \end{aligned}$$

Here we assume $i = 1$ and with all markets identical, $y_t = \sum_i y_t^i$ and $p_t = \sum_i p_t^i$. Also $\beta = b(1-\theta)$.

- Thus output fluctuations can occur if the island's price deviates from the expectation of the general price. This is the Lucas Supply Curve.

- Now we need a demand side to close the model. Consider the following relationship:

$$M_t V_t = P_t Y_t,$$

or, the nominal money supply times the velocity of money equals the price level times nominal GDP. Hence in logs:

$$m_t + v_t = p_t + y_t.$$

Note that usually $V_t = P_t Y_t / M_t$ is much larger than one because every additional dollar in the economy is spent many times over increasing the numerator more than the denominator.

- Consider the following money demand function:

$$M_t^d = k P_t Y_t.$$

So money demand is a constant fraction of real GDP. If money demand equals money supply:

$$k P_t Y_t = M_t^s,$$

Or,

$$\frac{1}{k} = \frac{P_t Y_t}{M_t^s} \equiv V_t.$$

So this fraction of real GDP that agent's demand is equal to the inverse of the velocity of money (Quantity Theory of Money).

- Normalize so that $V_t = 1 \Rightarrow v_t = log(V_t) = 0$. Hence our relationship that we started with becomes:

$$m_t + \underbrace{v_t}_{0} = p_t + y_t.$$

- Hence we have a supply side and a demand side:

$$\text{Supply:} \quad y_t = \beta(p_t - E[p_t | I_t]).$$

$$\text{Demand:} \quad y_t = m_t - p_t.$$

16 Lecture 16: March 30, 2005

16.1 More on Lucas Islands

- Recall our supply and demand equations from last time:

$$\text{Supply:} \quad y_t = \beta(p_t - E[p_t|I_t]), \ \beta = b(1-\theta).$$

$$\text{Demand:} \quad y_t = m_t - p_t.$$

Thus output fluctuations are driven by price expectational errors.

- Setting supply equal to demand:

$$\begin{aligned}
\beta(p_t - E[p_t|I_t]) &= m_t - p_t \\
E[\beta(p_t - E[p_t|I_t])|I_t] &= E[m_t - p_t|I_t] \\
\beta(E[p_t|I_t] - E[p_t|I_t]) &= E[m_t|I_t] - E[p_t|I_t] \\
E[m_t|I_t] &= E[p_t|I_t]
\end{aligned}$$

Thus there is a direct relationship between prices and money.

- Solving the system:

$$\begin{aligned}
\beta(p_t - E[p_t|I_t]) &= m_t - p_t \\
p_t(\beta + 1) &= m_t + \beta E[m_t|I_t] \\
p_t &= \frac{1}{1+\beta}(\beta E[m_t|I_t] + m_t)
\end{aligned}$$

And:

$$\begin{aligned}
y_t &= m_t - p_t \\
&= m_t - \frac{1}{1+\beta}(\beta E[m_t|I_t] + m_t) \\
&= -\frac{\beta}{1+\beta}E[m_t|I_t] + m_t(1 - \frac{1}{1+\beta}) \\
&= -\frac{\beta}{1+\beta}E[m_t|I_t] + m_t(\frac{\beta}{1+\beta}) \\
y_t &= \frac{\beta}{1+\beta}(m_t - E[m_t|I_t])
\end{aligned}$$

- Thus Lucas's point is that only unanticipated money matters! People only observe their own prices so they don't observe inflation directly. However, over time, they see that their price expectations are wrong so this drives output fluctuations.

- Thus the sequencing would be as follows: Suppose the fed cut rates so $m_t \uparrow$. This causes $p_t \uparrow$, but there is a signal extraction problem so $p_t^i \uparrow$ but the firms think it could be coming from $z_t^i \uparrow$. Thus $y_t^i \uparrow$ and hence $y_t \uparrow$.
- Thus any systematic changes in the money supply should not matter. If the fed annouces a rate cut, then $m_t \uparrow$ AND $E[m_t|I_t] \uparrow$, so while $p_t \uparrow$, $\Delta y_t = 0$.
- There are other versions of this type of model including when the factor markets operate with this type of price expectations.
- The data show that even announced policy actions do matter. There are three reasons they might matter:
  - (1) Credibility, Expectations Formation, and Learning. Do agents really know the model? Are they really that smart? Are they rational? There could also be uncertainty about what type of policy maker you are dealing with. There might even be an incentive for a policy maker to deceive people. By announcing that he will be tight with money drives down expectations of m_t. If he is then loose with money, from the y_t equation above, we should get good growth.
  - (2) Price and Wage Stickiness. Are they perfectly flexible? Probably not. If there is a slow adjustment process, we might get anticipated effects. Usually wages and prices are adjusted (renegotiated) every year and every 6 months respectively.
  - (3) Credit versus Money View of Monetary Policy. The Lucas Island model relies on the money view: money is all that matters. In reality, financial markets are more complicated and what really matters is how much credit is available in the economy.

Tedious Algebra

- Recall we had our price expectations equation from last time:

$$E[p_t|I_t^i] = (1-\theta)E[p_t|I_t] + \theta p_t^i,$$

  with,

$$\theta = \frac{\sigma_p^2}{\sigma_p^2 + \sigma_z^2},$$
$$I_t^i = \{I_t, p_t^i\}.$$

  Where did this come from?

- Some preliminaries. Note $f(y|x) = f(x,y)/f(x)$, so:

$$E[y|x] = \int_{-\infty}^{\infty} y f(y|x) dy = \int_{-\infty}^{\infty} y f(x,y)/f(x) dy.$$

  Suppose we consider distributions of y and x which induce conditional expectations:

$$E[y|x] = a + bx.$$

So conditional expectations are linear. This isn't that strong of an assumption since normal distributions have this property. Hence:

$$\begin{aligned}
\int_{-\infty}^{\infty} yf(x,y)/f(x)dy &= a+bx \\
\int_{-\infty}^{\infty} yf(x,y)dy &= (a+bx)f(x) \\
\text{Integrate over } x & \\
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x,y)dydx &= \int_{-\infty}^{\infty}(a+bx)f(x)dx \\
\int_{-\infty}^{\infty} yf(y)dy &= a+b\int_{-\infty}^{\infty} xf(x)dx \\
E[y] &= a+bE[x] \quad (*) \\
\text{Mult } x \text{ and Integrate over } x & \\
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dydx &= \int_{-\infty}^{\infty}(ax+bx^2)f(x)dx \\
E[xy] &= a\int_{-\infty}^{\infty} xf(x)dx + b\int_{-\infty}^{\infty} x^2f(x)dx \\
E[xy] &= aE[x]+bE[x^2] \quad (**)
\end{aligned}$$

Equations (*) and (**) are the normal equations of the system. Denote $E[y] = \mu_y$, $E[x] = \mu_x$, $Var(x) = \sigma_x^2$, etc. Thus, our equation are:

$$\mu_y = a + b\mu_x.$$

$$\sigma_{xy} + \mu_x\mu_y = a\mu_x + b(\sigma_x^2 + \mu_x^2).$$

So,

$$\sigma_{xy} + \mu_x(a + b\mu_x) = a\mu_x + b(\sigma_x^2 + \mu_x^2).$$

$$\sigma_{xy} + b\mu_x^2 = b\sigma_x^2 + b\mu_x^2.$$

$$\sigma_{xy} = b\sigma_x^2.$$

$$b = \frac{\sigma_{xy}}{\sigma_x^2}.$$

And,

$$\mu_y = a + \frac{\sigma_{xy}}{\sigma_x^2}\mu_x.$$

$$a = \mu_y - \frac{\sigma_{xy}}{\sigma_x^2}\mu_x.$$

- To go further, we need an additional fact which will be proved in class with Fabiano. In general:

$$E[y|\Omega, x] = E[y|\Omega] + E[y - E(y|\Omega)|x - E(x|\Omega)],$$

or in our case:

$$E[p_t|I_t, p_t^i] = E[p_t|I_t] + E[\underbrace{p_t - E(p_t|I_t)}_{``y"}\,|\,\underbrace{p_t^i - E(p_t|I_t)}_{``x"}].$$

Note that $E[p_t^i|I_t] = E[p_t|I_t]$. We need this result because we are conditioning on two items, not just one. So we can split the expectation on two things into two single conditioning expectations.

- Denote "x" and "y" as above. Denote:

$$\epsilon_t = p_t - E[p_t|I_t] \sim N(0, \sigma_p^2).$$

So we denote this deviation from expectations which must be distributed normally with mean 0 and the variance of p. Also recall:

$$p_t^i = p_t + z_t^i.$$

Thus, $\epsilon_t = ``y"$ and $\epsilon_t + z_t^i = ``x"$. Thus, $\mu_y = \mu_x = 0$ since ϵ and z are mean zero. Hence from our equations above,

$$\begin{aligned} a &= 0 \\ b &= \frac{\sigma_{xy}}{\sigma_x^2} \\ &= \frac{E[\epsilon_t(\epsilon_t + z_t^i)]}{E[(\epsilon_t + z_t^i)^2]} \\ &= \frac{E[\epsilon_t^2] + E[\epsilon_t z_t^i]}{E[\epsilon_t^2] + E[z_t^{i2}]} \\ &= \frac{\sigma_p^2 + 0}{\sigma_p^2 + \sigma_z^2} \\ &= \frac{\sigma_p^2}{\sigma_p^2 + \sigma_z^2} = \theta. \end{aligned}$$

Thus,

$$E[``y"|``x"] = \theta[p_t^i - E(p_t|I_t)].$$

So plugging this into the above expression yields:

$$E[p_t|I_t, p_t^i] = E[p_t|I_t] + E[\underbrace{p_t - E(p_t|I_t)}_{``y"}\,|\,\underbrace{p_t^i - E(p_t|I_t)}_{``x"}].$$

$$E[p_t|I_t^i] = E[p_t|I_t] + \theta[p_t^i - E(p_t|I_t)].$$

$$E[p_t|I_t^i] = (1-\theta)E[p_t|I_t] + \theta p_t^i.$$

And we're done.

- Finally, note that in this model, anticipated monetary policy moves do not matter. This is because the model is forward looking. We could also have the adaptive expectations model:

$$p_t^e = p_{t-1}^e + \lambda(p_{t-1} - p_{t-1}^e).$$

If we don't assume that agents "know" the model, then this backward looking behavior might be the best we can do. Here anticipated monetary policy changes might have an effect because even though agents know what's coming, they don't know exactly how the price or money changes will effect their market or level of output.

17 Lecture 17: April 4, 2005

17.1 Summarizing Lucas Islands

- The theme so far has been: "why does money matter?" In general, we have said it is expectation errors about inflation or prices. We have two types of models regarding expectations:
  - (1) Rational (Forward Looking) Expectations. Agents know the model, the distributions of underlying shocks, and the policy maker's reaction functions. In this setting, only unanticipated money matters. As in the simple Lucas Island model, we also do NOT get persistence in output fluctutations.
  - (2) Adaptive (Backward Looking) Expectations. Agents do not take all current information into account. We had the model:

    $$p_t^e = \lambda p_{t-1} + (1-\lambda)p_{t-1}^e.$$

    This yields an output equation:

    $$y_t = \beta(p_t - p_t^e).$$

    Thus both anticipated and unanticipated money matters. So if the fed annouces a cut in interest rates, agents don't take this completely into account so output is effected. It could be that the agents do not understand the model or the economy very well. This type of expectations DO yield persistence. Future values of output are effected by today's shocks and the amount of persistence depends on how quickly expectations adjust.

- So the key point is that we need adaptive expectations to get persistence in output. If the information set of the agent contains all current (and past) information, we cannot have persistence. So the propagation of the shocks is essential to get persistence into the model.

- Lucas (1975) introduced capital and serially correlated z_t^i's. Since capital is a durable which pays dividends in future periods, if we get a shock that effects today's level of the capital shock, it raises or lowers output in the future. So given:

  $$p_t^i = p_t + z_t^i.$$

  If agents see p_t^i go up, they think that some part of that increase is idiosyncratic and some part is general. Thus the agent should accumulate more capital if there is persistence in the idiosyncratic component. But introducing capital makes the model fairly complicated. In general, Lucas posed a problem for economists: "We need to get serial correlation in output without introducing serial correlation in the shocks."

- So this is heading towards the Long and Plosser model. Recall we can get persistence directly through shocks or the propagation of shocks. We might have real shocks like technological shocks, cost shocks, or raw material shocks (oil).

- Consider technological shocks. Consider a production function:

$$y_t = A_t F(x_t).$$

A_t may change over time in a stochastic fashion. But how can this lead to "negative growth"? This is a hard story to tell since we usually think that technological shocks make us better off. Any downward shock would be short-lived because presumably we would be able to still use the old technology. So Long & Plosser and Kydland & Prescott both study this idea. Both try to explain the real business cycle through fluctuations in A_t.

- However, trying to estimate A_t is hard due to the Solow residual. Suppose we try to estimated a cobb-douglas production function by running a regression (in logs):

$$y_t = a_t + \alpha k_t + \beta l_t.$$

Our estimate of $\hat{a}_t$ has been shown to follow y_t very closely, see G-17.1. So maybe it really is a_t that is driving recessions. However, there are three problems:

  - (1) Factor Utilization. Since utilization of labor and capital will also change in slump years, we may get stochastic coefficients.
  - (2) Restructuring and Reallocation. We usually run this regression with annual aggregate US data. Since factors can move from one sector to another, we are not controlling for this.
  - (3) Mismeasurment. Since we assume perfectly competitive markets (workers paid their marginal product for example), we are missing the mark-up associated with these markets not actually being competitive.

- So we will now outline the model of Long and Plosser who try to explain RBCs with technological shocks.

17.2 Long and Plosser Model

- We have a representative agent problem who is both the firm and worker. The agent consumes and produces in several sectors.

- Utility:

$$U = Max\ E[\sum_{t=0}^{\infty} \beta^t u(C_t, Z_t)].$$

Here C_t is an $Nx1$ vector of consumption in N goods. Z_t is leasure.

- Technology:

$$Y_{t+1} = F(L_t, X_t; \lambda_{t+1}).$$

Here Y_{t+1} is an $Nx1$ vector of outputs, L_t is an $Nx1$ vector of labor inputs in each of the N sectors, X_t is an NxN matrix of intermediate goods (so X_{ijt} is the amount of good j used to produce good i in time t), and λ_{t+1} is a vector of technology shocks. The key things here are that production takes time (inputs today produce output tomorrow) and goods produce goods (there will be trade off between consuming a good and using it to produce something else).

- Resource Constraints.

$$H = Z_t + \sum_{i=1}^{N} L_{it},$$

where H is the total time available, Z_t is again leisure, and L_{it} is labor time spent in sector i. We also have a production constraint:

$$Y_{jt} = C_{jt} + \sum_{i=1}^{N} X_{ijt}.$$

So output today can either be consumed or used to produce something for tomorrow.

- Model assumptions. Utility:

$$u(C_t, Z_t) = \theta_0 log(Z_t) + \sum_{i=1}^{N} \theta_i log(C_{it}).$$

Production:

$$Y_{i,t+1} = \lambda_{i,t+1} L_{it}^{b_i} \prod_{j=1}^{N} X_{ijt}^{a_{ij}}.$$

- Thus the value function looks like:

$$V(S_t) = Max\ \{u(C_t, Z_t) + \beta E[V(S_{t+1}|S_t)]\}.$$

Where the states are: $S_t = \{Y_t, \lambda_t\}$.

- As we show on the homework, we guess a value function of the form:

$$V(S_t) = \sum_{i=1}^{N} \gamma_i ln(Y_{it}) + J(\lambda_t) + K,$$

where K is a collection of constants. With:

$$\gamma_j = \theta_j + \beta \sum_{i=1}^{N} \gamma_i a_{ij},$$

$$J(\lambda_t) = \beta E[\sum_{i=1}^{N} \gamma_i ln(\lambda_{i,t+1}) + J(\lambda_{t+1}|\lambda_t)].$$

- This yields the following 4 equations:

$$C_{it} = \left(\frac{\theta_i}{\gamma_i}\right) Y_{it}. \quad (1)$$

$$X_{ijt} = \left(\frac{\beta\gamma_i a_{ij}}{\gamma_j}\right) Y_{jt}. \quad (2)$$

$$Z_t = \left(\frac{\theta_0}{\theta_0 + \beta\sum_{i=1}^{N}\gamma_i b_i}\right) H. \quad (3)$$

$$L_{it} = \left(\frac{\beta\gamma_i b_i}{\theta_0 + \beta\sum_{j=1}^{N}\gamma_j b_j}\right) H. \quad (4)$$

- We will work with these more next time, but (1) shows that if θ_i is high relative to γ_i, meaning that good i is more valuable in terms of consumption than using it to produce other goods, then eat it. Equation (2) shows that we should use current output of good j for production of good i if both good i is valuable (γ_i) and good j is valuable in the production of good i (a_{ij}).

- More next time.

18 Lecture 18: April 6, 2005

18.1 More on Long and Plosser

- Recall our 4 equations from last time that solve the model:

$$C_{it} = \left(\frac{\theta_i}{\gamma_i}\right) Y_{it}. \quad (1)$$

$$X_{ijt} = \left(\frac{\beta\gamma_i a_{ij}}{\gamma_j}\right) Y_{jt}. \quad (2)$$

$$Z_t = \left(\frac{\theta_0}{\theta_0 + \beta\sum_{i=1}^{N}\gamma_i b_i}\right) H. \quad (3)$$

$$L_{it} = \left(\frac{\beta\gamma_i b_i}{\theta_0 + \beta\sum_{j=1}^{N}\gamma_j b_j}\right) H. \quad (4)$$

- Key results: From (1) and (2), we see that the relative production versus consumption value of an intermediate good is important. The other result involves the available resources ($H + Y_t$). If all of a sudden, the agent has more H, he will devote some to labor and some to leisure. More Y_t will be allocated between consumption and intermediate goods.

- Now recall the production function:

$$Y_{i,t+1} = \lambda_{i,t+1} L_{it}^{b_i} \prod_{j=1}^{N} X_{ijt}^{a_{ij}}.$$

  Take logs:

$$y_{i,t+1} = log(\lambda_{i,t+1}) + b_i log(L_{it}) + \sum_{j=1}^{N}[a_{ij} log(X_{ijt})].$$

  Substitute in:

$$y_{i,t+1} = log(\lambda_{i,t+1}) + b_i log\left(\left(\frac{\beta\gamma_i b_i}{\theta_0 + \beta\sum_{j=1}^{N}\gamma_j b_j}\right) H\right) + \sum_{j=1}^{N}\left[a_{ij} log\left(\left(\frac{\beta\gamma_i a_{ij}}{\gamma_j}\right) Y_{jt}\right)\right].$$

  Or,

$$y_{i,t+1} = Ay_t + k + \eta_{t+1},$$

  a Vector Auto Regression (VAR), as output of sector i in period $t+1$ is related to output in all other sectors in period t. Clearly the a_{ij} coefficients are going to be crucial to the analysis.

- So what's happening here? Suppose there is a shock in period t: $\lambda_{it} \uparrow$. This could be sectoral or general. Via the production function, $Y_{it} \uparrow$. This causes $C_{it} \uparrow$ via (1) and $X_{jit} \uparrow$ via (2). But since X_{jit} goes up, this means that next period's output also rises: $Y_{i,t+1} \uparrow$. This cause $C_{j,t+1} \uparrow$ and $X_{kj,t+1} \uparrow$, and so on. So we get persistence in output movements due to two forces: **production takes time and goods produce goods**.

- The criticism of the model is that economy wide adverse technology shocks are hard to justify. Maybe a flood across the entire country? Usually the story is much easier to tell at a micro level.

- Also notice that the labor/leisure tradeoff is not entering the RBC. The reason is our functional form of the utility function: income and substitution effects exactly cancel each other out. In a more general setting, you would not have this result, but the solution is more complicated.

- So when we run the VAR above, we find that sectoral growth rates do tend to move together: Sectoral comovements at business cycle frequencies. For example, for the manufacturing and service sector:

$$\rho_{y_M, y_S} = 0.63.$$

  Where y_M is the growth rate of output in the manufacturing sector. Why do we get these comovements?

  - (1) Common shocks – enough shocks are not sector specific that the correlation is high.
  - (2) Sectoral Complementarities which are all forms of "External Increasing Returns" that include:
    * (a) Input/Output process – via the X matrix.
    * (b) Technological externalities – knowledge spillovers.
    * (c) Demand externalities/spillovers – a slump in one sector will influence another sector through decreased demand for their products.

- So what is the empirical evidence on co-movement? As we said, we find $\rho_{y_i y_j} > 0$ for many sectors. However, when we estimate the VAR, we find that common shocks only account for a small part of the co-movement. $\rho_{\eta_i \eta_j} = 0.08$ for the manufacturing versus the service sector. So the other component, sectoral complementarities, must be driving the co-movement. Shea has a paper where he found that this input/output matrix was essential to the comovement.

- Punchline: we get persistence through goods producing goods and production takes time.

18.2 Production Smoothing/Buffer Stock Model (Blinder/Fischer)

- Suppose there is a slump in aggregate demand: $y^d \downarrow$. Firms, not wanting to assume this is going to be a long recession, just let inventories rise instead of cutting back production. Eventually they realize that the slump is not short-lived, so they cut production, $Y \downarrow$. When the recession ends, production doesn't immediately recover. It only recovers when excess inventories have been reduced. See G-18.1.

- In this model we assume convex costs. See G-18.2. Firms could produce at A, or they could save costs by producing at B and use inventories to cover periods of low demand. Because of the shape of the cost curve, marginal cost of production is very high if they had to produce up near Y_H, so this makes inventories attractive. Firms have an incentive to smooth production.

- Firm's face downward sloping demand:

$$\underbrace{p_t}_{firm's\ nominal\ price} = \overbrace{v_t}^{idiosyncratic\ shock} \underbrace{P_t}_{general\ price\ level} D(\overbrace{X_t}^{sales}).$$

  This is similar to the Lucas Island model in that firms will not observe the general price level but rather only their own price. They will have to extract the general price level using their signal, their own price.

- More next time.

19 Lecture 19: April 11, 2005

19.1 More on Production Smoothing/Buffer Stock Inventory Model

- Recall we have been getting persistence into our models by introducing a durable item which is effected by a non-serially correlated shock. This could be capital, sectors where production takes time, or inventories.

- In the inventory model, we have the following key equations:

$$\text{Demand Schedule: } p_t = v_t P_t D(X_t),$$

  where p_t is the nominal price of the firm, v_t is an idiosyncratic shock, P_t is the general price level and X_t is sales. Usually the firm will not observe P_t but only p_t so it faces a signal extraction problem. Firm's face costs:

$$\text{Cost Schedule: } c_t = P_t c(Y_t),\ c' > 0, c'' > 0$$

  so we have nominal costs equal the general price level times real costs. The convexity of this cost curve is key to the model. See the graph from last lecture. We have the dynamics of inventories for the firm:

$$\text{Inventories: } N_{t+1} = N_t + Y_t - X_t,$$

  so next periods inventory stock equals today's inventory plus production minus sales. Since this allows $Y_t \neq X_t$, we can get production smoothing. We can also express firm revenues:

$$\text{Revenues: } R(X_t, v_t) = \frac{p_t X_t}{P_t} = \frac{v_t P_t D(X_t) X_t}{P_t} = v_t D(X_t) X_t,\ R_x > 0, R_{xx} < 0,$$

  so real revenues are a function of sales and the shock term. Finally profits:

$$\text{Profits: } \pi_t = R(X_t, v_t) - c(Y_t) - \frac{B(N_{t+1})}{1+r}.$$

  Here the $B(\cdot)$ function is the cost of maintaining inventories and is shown in G-19.1. We think of symmetric costs of holding positive inventories (excess stock) or negative inventories (unfilled orders).

- As will be shown on the problem set, we set up the Bellman's equation as follows:

$$J(N_0) = Max\ \{\pi_0 + E_0[\frac{J(N_1)}{1+r}]\}.$$

  Note that today's inventories (time zero) are the key state to this DPP.

- Solution. Optimality implies two conditions:

- (1) $R_x = c_Y$, or marginal revenue (with respect to sales) equals marginal cost (with respect to output).
- (2) $E[\frac{J'(N_1)}{1+r}] = c' + \frac{B'}{1+r}$. So on the RHS, we have the marginal costs of production and of holding inventories.

- So instead of assuming functional forms and finding a closed form solution, we will just do comparative statics on the model.

- Key Results.

$$
\begin{array}{ccccccccccc}
0 & < & \frac{\partial X_0}{\partial N_0} & < & 1, & \frac{\partial X_0}{\partial v_0} & > & 0, & \frac{\partial X_0}{\partial(1+r)} & > & 0 \\
0 & < & \frac{\partial N_1}{\partial N_0} & < & 1, & \frac{\partial N_1}{\partial v_0} & < & 0, & \frac{\partial N_1}{\partial(1+r)} & < & 0 \\
-1 & < & \frac{\partial Y_0}{\partial N_0} & < & 0, & \frac{\partial Y_0}{\partial v_0} & > & 0, & \frac{\partial Y_0}{\partial(1+r)} & < & 0
\end{array}
$$

- So suppose there is a shock and $v_0 \uparrow$. By the (2,2) and (3,2) terms, current output rises and next period inventories fall. So the firm is responding to the increase in aggregate demand by increasing output a little and also reducing their inventory stock. If you update term (3,1), we see that next period, output will rise again to replenish some of the inventory stock that was depleted last period. Thus output is rising in both periods (and indeed in future periods, but less and less), so we get persistence.

- The key to all these comparative statics is the shape of the real costs curve, namely $c'' > 0$. We also can show that we get the same result even if P_t is unknown and must be extracted. A firm will think that at least part of a nominal price jump is do to their idiosyncratic component so they will increase production today AND reduce inventories today. The latter affects output tomorrow (positively). In reality, a nominal money shock (even for just one period), can cause this exact reaction so we have an answer to Lucas' challenge of getting output persistence out of serially UNcorrelated nominal shocks.

- The punchline of the model is a prediction:

$$Var(Y_t) < Var(X_t).$$

  So by using inventories to buffer, we should see less variation in output than in sales.

- As with all things Macro, the data show the exact OPPOSITE! Using empirical data on inventories, economists find:

$$Var(Y_t) > Var(X_t).$$

- So what are the possible explanations for this? It could be either that the data is wrong or the model itself is wrong. Consider the two possibilities in turn.
- Data Problems.
  - (1) LIFO/FIFO. Most inventory data is in value form and the price by which a firm deflates the nominal amounts and if they use a First-In-First-Out or a Last-In-Last-Out accounting method is crucial.
  - (2) Physical unit data unavailable. This is what we really want and with say, the car industry, it's easy to count cars, but with many industries, this is simply impossible to find.
  - (3) Plant Level Price Heterogeneity. We find that even within industry, prices can be fairly different so this introduces a lot of noise.
- Model Problems. Alternatives:
  - (1) Non-Convexities. See G-19.2. When we actually estimate the cost structure, it is clearly NOT convex. There are fixed costs associated with setting up a plant and often times (particularly in the auto sector), it makes sense to operate at a high level on some days and shut down completely on others. This setup lends itself towards production bunching, not smoothing. Produce a lot once in a while but then shut down because the fixed costs are too high to just produce a little all the time. When we look at micro level data on the variability in output versus sales, we see something like G-19.3. The bunchers are located on the right (car makers for example) and the smoothers are located around the lower mode. There doesn't seem to be much in the data about what type of sectors or firms are located at each mode.
  - (2) Cost/Technology Shocks. See G-19.4. As in Long and Plosser, if business cycles are driven by REAL shocks, we might have a cost schedule that is subject to shocks. In this setting, firms produce a lot when costs are low and little when costs are high, regardless of sales levels. This might drive up $Var(Y_t)$ as we see in the data.
  - (3) Stock-out Avoidance and Serially Correlated Shocks. See model of Kahn and Bils. The cost minimization in their model is:

    $$Min\ E[\sum_{i=0}^{\infty} \beta^i (\frac{a}{2}(Y_{t+i} - \underbrace{U_{t+i}}_{cost\ shocks})^2 + \frac{b}{2}(Y_{t+i} + I_{t+i-1} - I^*_{t+I})^2)].$$

    And the authors assume $I^*_t = E[X_t] + k$, so inventories should be kept close to expected sales. The idea is that if you have a situation where firms cannot supply the market in a period of high demand, they lose a lot of business both today and in the future. So if they expect sales to grow in the future, there might be a period of high production to prepare for the possibility of high demand ($\uparrow Var(Y_t)$).
- So we have a situation where inventory data is highly volatile and hard to come by. When we do have it, it's usually mismeasured. Physical unit data would be better

but that's even more rare. There was even something in the news about Just-In-Time Inventory accounting (used first by the Japanese car makers) that was inducing US firms to reduce inventory to sales ratios. Holding on to inventories was too expensive and was costing the US in terms of competition with foreign firms. Over time, we've basically found that this effect really wasn't there to begin with and it was all just mismeasurement. Beautiful.

20 Lecture 20: April 13, 2005

20.1 Adjustment Cost Models

- Adjustment cost models are sort of a black box that tries to explain why, in the aggregate, some factors are slow to adjust following a shock. See G-20.1. In the aggregate, when the shock hits, it seems the factor is affected in a fairly smooth way, both the initial downturn and the recovery.

- Consider a standard profit function:

$$\pi = p_t f(L_t) - w_t L_t \Longrightarrow f'(L_t) = w_t/p_t.$$

  We could also add adjustment costs:

$$\pi = p_t f(L_t) - w_t L_t - C(\Delta L_t).$$

  The $C(\cdot)$ function reflects search costs of hiring, training and firing workers, as well as changes in the scope or scale of operations.

Sargent Model

- Sargent has a rather complicated general equilibrium model where a social planner maximizes:

$$Max\ E_t\left[\sum_{j=0}^{\infty} b^j\{U_0 c_{t+j} - (\delta_0 + \underbrace{\epsilon_{t+j}}_{taste\ shocks})\underbrace{n_{t+j}}_{labor} - \frac{\delta_1}{2}n_{t+j}^2 - \frac{\delta_2}{2}(n_{t+j} + \gamma n_{t+j-1})^2\right],$$

  subject to:

$$c_{t+j} = (f_0 + a_{0j})n_{t+j} - \frac{f_1}{2}n_{t+j}^2 - \underbrace{\frac{d}{2}(n_{t+j} - n_{t+j-1})^2}_{Adjustment\ Costs}.$$

  So the planner maximizes utility subject to the constraint that involves costs to adjusting labor depending on the parameter, d.

- Consider the more simple 2-period partial equilibrium problem where the firm maximizes:

$$Max \sum_{i=t}^{t+1} \beta^i[\underbrace{f_{0i} n_i - \frac{f_{1i}}{2}n_i^2}_{quadratic\ production} - w_i n_i - \frac{d}{2}(n_i - n_{i-1})^2].$$

  Note that if $d = 0$, ie there are NO adjustment costs:

$$n_t = \frac{f_{0t} - w_t}{f_{1t}}.$$

This is what we would like to set our demand for labor equal to. When $d > 0$, however:

$$n_t = \frac{f_{0t} - w_t + dn_{t-1} + \beta d(\frac{f_{0,t+1} - w_{t+1}}{f_{1,t+1} + d})}{f_{1t} + d + \beta d + \frac{\beta d^2}{f_{1,t+1} + d}}.$$

So we see the no adjustment cost term in there, but also a bunch of other terms involving d. In general we have:

$$n_t = \alpha_0 + \alpha_1 n_{t-1} + \alpha_2 x_t + \eta_t,$$

where x_t is a vector of other exogeneous variables and η_t is a shock term. We can see that $\alpha_1 < 1$ but it depends crucially on the level of d. If adjustment costs rise, then α_1 will also rise as the effect of last period's labor demand will be more important in today's choice of labor.

- Thus, convex adjustment costs (quadratic) yield persistence in the factors. The speed of adjustment, α_1, depends critically on d.
- When we estimate α_1 using aggregate data, we get $\hat{\alpha}_1 \approx 0.9$. This seems implausibly high. The adjustment process would then be TOO slow compared with what we see. The problem is in estimating α_1. Since there are both wage and productivity shocks also present, η_t will be serially correlated and it's hard to distinguish between persistence due to shocks versus persistence due to the propagation (adjustment cost story).
- We have also tried looking at micro level data. See G-20.2. In the micro level data, we don't get the smooth adjustment processes but rather a large jump in (say) the demand for labor when the shock hits. In fact, we see that the distribution of the growth rate of labor demand is centered around a spike around 0. About 17% of firms in the sample have essentially no change in labor demand at a quarterly frequency. So there is a lot of inertial behavior.
- So what is the alternative at the micro level. We could try NON-convex adjustment costs. Consider a firm with profit function $\pi(A_t, L_t)$ where A_t are technology and wage shocks. In periods of no adjustment the firm has a value function:

$$V^{no\ adj}(L_{t-1}) = \pi(A_t, L_{t-1}) + \beta E[V(L_{t-1})],$$

while when adjustments are made:

$$V^{adj}(L_{t-1}) = \pi(A_t, L_t) - F + \beta E[V(L_t)].$$

Where F is a fixed cost of adjustment. So in the first case, the firm faces the same labor each period. Thus the problem of the firm is:

$$V(L_{t-1}) = Max\ \{V^{no\ adj}(L_{t-1}), V^{adj}(L_{t-1})\}.$$

21 Lecture 21: April 18, 2005

21.1 More on Adjustment Costs

- When we look at macro level aggregate data of employment we see a picture like G-21.1. The level of employment over time is relatively smooth.

- We can get a picture like this using convex adjustment costs where the model yields:

$$\Delta L_t = \lambda(L_t^* - L_{t-1}) = \lambda z_t.$$

  Each period, we get partial adjustment to the target rate. However, when we estimate the model (rewritten):

$$L_t = \lambda L_t^* + (1-\lambda)L_{t-1}.$$

  We get an estimate for $1-\lambda$ of 0.9. This is actually TOO high! This would mean that we actually have too much persistence and things are really slow to adjust. Therefore the model is not plausible.

- When we move to the micro level data we see something like in G-21.2, a Kurtotic distribution with a spike at 0 and heavy tails. Therefore, we either get no adjustment or a LOT of adjustment. What kind of model generates this? A non-convex adjustment cost model. One of these is the Generalized (S,S) model by Caballero and Engel. The key result of this type of model is:

$$\Delta L_{et} = A(L_{et}^* - L_{e,t-1}) = A(z_{et}).$$

  In this case, we might expect a picture like G-21.3. There is some range of inaction for the firms so you need to hit some threshold before they adjust. One simple non-convexity would be a "fixed-cost" of adjustment so that even a small change costs something relatively large.

- So this micro story might work, but how do we aggregate to the macro economy? Consider:

$$\Delta L_t = \int zA(z)f(z)dz,$$

  where $A(z)$ is the adjustment function. The idea is that we need to integrate over all firms but include the cross sectional distribution, $f(z)$. The shape and location of this distribution is key. See G-21.4. An aggregate shock that moves us from $f(z_0)$ to $f(z_1)$ will mean that a bunch of firms that had not reached their threshold will now decide it is worthwhile to adjust. Clearly, the heaviness of the tails and the location of the threshold levels are crucial in determining how much adjustment happens following a shock.

- As an example, if $A(z) = \lambda$, then $\Delta L_t = \lambda(L_t^* - L_{t-1})$, but making $A(z)$ linear is probably not a good assumption. If $A(z)$ is highly nonlinear, the nonlinearity at the micro level will interact with the cross-sectional distribution, $f(z)$, so the aggregate response to even an aggregate shock will be complicated.

- Consider a further example where we move from $f(z_1)$ to $f(z_2)$. Now we get a LOT of adjustment! See G-21.5.
- Thus, with this model, we can either get almost no adjustment or a lot of adjustment. This is consistent with G-21.2. But what about the aggregate picture G-21.1? Well, the cross-sectional distribution is what reconciles the micro and macro level pictures. Since some shocks might affect some firms more than others, (and once a firm adjusts, they basically move to the center of the spike), the overall aggregate picture might look fairly smooth, as in G-21.1.

21.2 Another Non-Convex Adjustment Cost Model

- Consider a value function of a firm:

$$V(A, e_{-1}) = Max\ \{V^n(A, e_{-1}), V^a(A, e_{-1})\}.$$

Where the value functions for (n)o adjustment and (a)djustment are:

$$V^n(A, e_{-1}) = Max_{e,h}\ \{R(A, e_{-1}, h) - \omega(e_{-1}, h) + \beta E_{A'|A}[V(A', e_{-1})]\}.$$

$$V^a(A, e_{-1}) = Max_{e,h}\ \{\lambda R(A, e, h) - \omega(e, h) - F - \frac{\nu}{2}(e - e_{-1})^2 + \beta E_{A'|A}[V(A', e)]\}.$$

- So the firm chooses to adjust or not adjust depending on the values of these two functions. In the no adjustment value function, $R(\cdot)$ is the revenue function with arguments including a profit shock (A), last period's employment level (e), and hours worked by current employees (h). ω is the compensation function for levels of employment and hours worked. Note that in the adjustment value function, we now update the level of employment but include:
  - (1) Non-convex fixed cost of adjustment (F) and,
  - (2) Sargent's convex adjustment costs (quadratic term).
- Note also the λ parameter. This is sort of a disruption effect. Anytime the firm decides to adjust their level of employment, there is a period of time where they will probably have slightly lower productivity as they "learn" how to operate in their new environment. Thus $\lambda < 1$.
- Finally note that the hours variable in the model is like a buffer stock for the firm. Instead of immediately hiring or firing workers, they might simply use overtime hours or cut back workers to part time. This also provides some employment smoothing to the firm and avoids them having to pay the fixed (and convex) costs of adjustment.
- Closed form solutions to this problem are impossible but we can solve numerically using either Indirect Inference or Simulated Method of Moments. Consider the latter. We might discretize the state space, make some assumption on the parameters and estimate the model completely. We could then calculate a vector of simulated moments:

$$M^s(\theta),$$

where θ might contain parameters of the compensation function and of the profit shock process. We then estimate sample moments from the actual data, $M^d(\theta)$ and minimize the following method of moments estimator:

$$Min_\theta \ (M^d(\theta) - M^s(\theta))w(M^d(\theta) - M^s(\theta))'.$$

- Some notes on simulated methods of moments:
  - (1) We need at least as many moments as we have parameters to estimate.
  - (2) We need to choose moments that are both relevant (sensitive to parameter values) and that can be estimated precisely (low standard errors) in the data.
- One result out of all of this is:

$$\rho^{micro}_{\Delta h, \Delta e} = -0.3,$$
$$\rho^{macro}_{\Delta h, \Delta e} = +0.54.$$

  So after doing method of moments estimation, we find that at the micro level, the change in hours employed and the change in workers employed is negatively related while at the macro level, the sign is completely reversed. While this normally wouldn't be that surprising because we are doing macroeconomics and nothing like this ever works, we do have an explanation for this result.
- Consider a small shock at the micro level so $A \uparrow$. Due to the fixed costs of adjustment, $\Delta e = 0$ but $h \uparrow$ as firms use their buffer to respond the profit shock. Now consider a large profit shock so $A \Uparrow$. In this case $e \uparrow$ and $h \downarrow$ as firms realize that they can reduce the number of hours worked (less overtime pay, say) because it is now worthwhile to hire workers. The second result is what is driving the negative correlation at the micro level.
- At the macro level, since not all shocks effect all firms at the same time and in the same way, some react to shocks by increasing employment (ie, if their threshold has been crossed) and some may just increase hours worked by current employees. So $\uparrow e$ for some and $\uparrow h$ for others, hence the positive correlation at the macro level. I'm not convinced.

21.3 Role of Search Frictions

- Note that we have provided several reasons for persistence in macro models:
  - (1) Adaptive expectations.
  - (2) Capital accumulation (durable goods story).
  - (3) Inventories.
  - (4) Adjustment Costs.
  - (5) Production takes time and goods produce goods.

- Another way to get persistence in through employment search frictions (Pissarides). Consider the following model:

$$\Delta u_t = \phi_t - \pi_t u_{t-1},$$

where u_t is unemployment, ϕ_t is the inflow rate into unemployment either from employed or from Not in the Labor Force (NLF), and π_t is the rate if outflow from unemployment to employment (either from being unemployed or from NLF).

- Note that π_t for the US is around 0.5 per month. So 50 percent of unemployed workers get back in the labor force every month (very high on global standards). The question is why isn't this number equal to 1? The answer is search frictions. It takes time to find a job. The matching process is slow.

- Rewrite the model:

$$u_t = \phi_t + (1 - \pi_t)u_{t-1}.$$

This is sort of like an AR(1) model and for $1 - \pi_t < 1$, we can generate persistence in employment (or unemployment). See G-21.6. Given a shock at t_0, the unemployment rate will not immediately return to its "natural" level, but rather take some time to come back down. The adjustment process depends crucially on π_t, the measure of search frictions.

22 Lecture 22: April 20, 2005

22.1 More on Search Frictions

- Recall the flow approach to unemployment dynamics:

$$\Delta u_t = \phi_t - \pi_t u_{t-1},$$

  where ϕ_t is the flow in to unemployment and π_t is the flow out of unemployment (ie, the job finding rate).

- This model yields persistence because it takes time for people to find jobs.

- Even in good times, $\overline{\phi}_t > 0$, as people are still moving in and out of employment. Thus, in the US, equilibrium rates are:

$$\overline{\phi}_t \approx 0.03,$$
$$\overline{\pi}_t \approx 0.5.$$

  So the equilibrium rate (or natural rate) of unemployment is $\frac{\overline{\phi}}{\overline{\pi}} \approx 6$ percent, so this model does pretty well for the US.

- In the data, we observe that when a recession hits, $\phi \uparrow$ and $\pi \downarrow$. Both of these make the jump larger and the recovery longer so they increase persistence. Also, when people lose their job and then find a new one, their new job is usually fairly fragile and much more likely to end then a long term position. Separations beget separations.

- So why do these flows change during recessions? It seems intuitive but it might be partially a result of markets not working effectively. If the labor market is simply slow to adjust, we may get persistence.

- Consider the outflow or escape rate from unemployment, π_t. We generally think that this is actually the weighted sum of escape rates of different types of people:

$$\pi_t = \sum_i \frac{s_{i,t-1}}{s_{t-1}} \pi_{it}.$$

  The types might be age, reason for unemployment - permanent or termporary, etc. We find the following:

$$\pi_{young} >> \pi_{old} \ \ (and\ \phi_{young} >> \phi_{old}).$$

$$\pi_{permanent} << \pi_{entrants}.$$

  So young people tend to re-enter the labor market more often than old, perhaps due to old people being more specific about job duties, geographic location, etc. When π falls in a recession, the mix of the unemployed changes. You get more slow-searchers, the old guys. If just the young, recent entrants to the labor market lose their jobs, they are usually quick to find new ones. When the older workers are searching, this really introduces a lot of persistence to the shock because the matching process is now much longer.

- An open question: Is this drop in π in a recession an equilibrium response or the result of a market imperfection. Open to debate.
- So, to recap, we found persistence in equilibrium models from two sources:
  - (1) Shocks. They must be real and serially correlated because nominal shocks will be incorporated into people's expectations so they don't generate persistence.
  - (2) Propagation. Capital, inventories, adjustment costs, search frictions, production takes time/goods produce goods.

  The punchline is that people are skeptical of the magnitude and persistence of the shocks so we really look towards propagation.
- Another propagation mechanism is wage/price stickiness. This is usually not put under the heading of equilibrium models, but another open question might be: Is the slowness of wage and price adjustment an equilibrium (optimal) response? It might be costly to adjust prices so there really is no friction, just optimal behavior.
- So in equilibrium models of persistence, we typically get persistence from some form of a "smoothing incentive." Agents do not want to react to a shock immediately but rather absorb the shock over time. These are all Dampening Models. In this situation, you need a really large and long lasting shock to match the data.
- So we move away from equilibrium models to a new New Keynesian setting where markets are not perfect and even nominal shocks can generate persistence through multiplier effects and market failures.

22.2 New Keynesian Macro Models

Nominal Rigidities - Akerlov and Yellen

- Consider the idea of Near Rationality: agents face costs of adjustment so it is NOT optimal to adjust your actions at every moment, but rather you tolerate small mistakes.
- See G-22.1. This is similar to the envelope theorem idea. You're optimizing at x^* and then there is a shock which makes your new optimal value something different. The loss from not adjusting can be expressed:

$$L(\epsilon) = f(x^*(\alpha_0 + \epsilon), \alpha_0 + \epsilon) - f(x^*(\alpha_0), \alpha_0 + \epsilon).$$

  So it's the difference between the new objective function evaluated at the optimal bundle and the new objective evaluated at your old bundle.
- Take a taylor series expansion of $L(\epsilon)$ around 0, we find:

$$L(\epsilon) \approx L(\epsilon)|_{\epsilon=0} + L'(\epsilon)|_{\epsilon=0}\epsilon + L''(\epsilon)|_{\epsilon=0}\frac{\epsilon^2}{2} + \ldots.$$

$$L(\epsilon) \approx \underbrace{L(0)}_{0} + L'(0)\epsilon + L''(0)\frac{\epsilon^2}{2} + \dots.$$

Consider the second term:

$$L'(\epsilon) = \underbrace{\frac{\partial f(x^*(\alpha_0+\epsilon), \alpha_0+\epsilon)}{\partial x^*}}_{=f_1=0} \underbrace{\frac{\partial x^*(\alpha_0+\epsilon)}{\partial \epsilon}}_{\neq 0} + \underbrace{f_2(x^*(\alpha_0+\epsilon), \alpha_0+\epsilon) - f_2(x^*(\alpha_0), \alpha_0+\epsilon)}_{=0 \ at \ \epsilon=0} = 0.$$

- So the cost of small mistakes/inertial behavior is only second order in the shocks.
- See G-22.2 for Mankiw's menu adjustment costs. We have a monopolist facing a downward sloping demand and initially charges prices p_0^* and sells q_0^*. Then there is a shift in demand out to D_1 and MR_1. If the monopolist does NOT adjust and charges p_0 and sells q_0 (no longer optimal in terms of profit), his profit is the area BCFG. Consumer surplus is ABC so total surplus is AFGC. If the monopolist adjusts to the optimal p_1^*, q_1^*, his new profit is DFIH and consumer surplus is ADH. Total surplus becomes AFIH. Thus, the loss from not adjusting is orange - green. The loss in social surplus is purple + orange. Thus the profit loss is second order (and therefore may not be worthwhile), but the social loss of not adjusting is first order since:

$$\text{Purple + Orange} >> \text{Orange - Green.}$$

23 Lecture 23: April 25, 2005

23.1 More on Menu Costs

- Key quote: "Though it may not be very costly to accept small errors, the effects of not adjusting on the aggregate economy could be significant."
- See homework for the result that the profit loss to the monopolist from not adjusting is second order in the shock to demand, while the social surplus lost from not adjusting is first order and relatively large.
- One big criticism of the Akerlof and Yellen paper is the assumption of linear costs. We tell a price inertia story, but it could be costs that are driving the rigidities.
- In the Mankiw paper, he introduces an adjustment cost, z, such that:

$$\text{Surplus Loss} > z > \text{Profit Loss}.$$

  Thus the agent will NOT adjust since the costs of doing so are larger than what he's losing in profits by not optimizing. However, it is still socially optimal to adjust. The conclusion is the same, but Mankiw just formalizes the adjustment cost.

Akerlov and Yellen's Model

- Consider a firm facing the following downward sloping sales/demand equation:

$$X = \left(\frac{p}{\overline{p}}\right)^{-\eta}\left(\frac{M}{\overline{p}}\right),$$

  where p is the price for the firm, M is the money supply and $\overline{p}$ is the general price level.

- Note that if $p = \overline{p}$, then $\overline{p}X = M$, ie, nominal income equals the money supply.
- Also assume the firm as a production function over labor (N) and effort (e):

$$X = (eN)^{\alpha},\ \alpha < 1,\ e = e(\omega).$$

  Thus, effort depends on the real wage, ω. This is just the efficiency wage idea as in Shapiro/Stiglitz. Assume $e' > 0$.

- Note in this problem, we have two types of market failures: the firm has monopoly power in the output good as well as some sort of monopsony power in the labor market with respect to this efficiency wage idea.
- Thus the firm's problem:

$$Max_{p,\omega}\ p * \left(\frac{p}{\overline{p}}\right)^{-\eta}\left(\frac{M}{\overline{p}}\right) - \omega\overline{p}N.$$

Note we maximize over ω, even though the firm doesn't choose the real wage. However, he chooses p, knows $\overline{p}$, and this implies the real wage.

- FOCs (Derive these!):

$$\frac{e'(\omega)\omega}{e} = 1.$$

$$p = kM, \quad k = \left(\frac{\eta\omega^*}{\alpha(\eta-1)e(\omega^*)}\right)^{\alpha/(1-\alpha)}.$$

This implies:

$$X = \frac{M}{p} = \frac{1}{k}.$$

$$N = \frac{X^{1/\alpha}}{e} = \frac{k^{-1/\alpha}}{e(\omega^*)}.$$

- So the key result is that both output/sales and the demand for labor are independent of M (neutrality of money).
- Now, we add a complication to the model. Assume a proportion, β, of the agents are full maximizers (they face no menu costs), and $1-\beta$ of the agents face menu costs, so we call them "Near Rational."
- Consider a shock from money level M_0 to $M_0(1+\epsilon)$.
- We get the following results (Derive these):
  - (1) (n)on-maximizers (same prices and wages as under M_0):

$$p^n = p_0.$$

$$\omega^n = \frac{w^n}{\overline{p}} = \frac{w_0}{\overline{p}}.$$

  - (2) (m)aximizers:

$$p^m = p_0(1+\epsilon)^\theta, \quad \theta = \frac{(1-\alpha)\alpha}{\beta(\eta/\alpha - \eta + 1) + (1-\beta)(1-\alpha)/\alpha}.$$

$$\omega^m = \omega^*.$$

This implies:

$$\overline{p} = (p^m)^\beta (p^n)^{1-\beta} = p_0(1+\epsilon)^{(1-\beta)\theta}.$$

And we also get the following results for sales and labor demand:

$$
\begin{aligned}
X^m &= \left((1+\epsilon)^{\beta\theta}\right)^{-\eta}\left(\frac{M_0}{p_0}(1+\epsilon)^{1-(1-\beta)\theta}\right) \\
X^n &= \left((1+\epsilon)^{(\beta-1)\theta}\right)^{-\eta}\left(\frac{M_0}{p_0}(1+\epsilon)^{1-(1-\beta)\theta}\right) \\
N^m &= \frac{\left[\left((1+\epsilon)^{\beta\theta}\right)^{-\eta}\left(\frac{M_0}{p_0}(1+\epsilon)^{1-(1-\beta)\theta}\right)\right]^{1/\alpha}}{e(\omega^*)} \\
N^n &= \frac{\left[\left((1+\epsilon)^{(\beta-1)\theta}\right)^{-\eta}\left(\frac{M_0}{p_0}(1+\epsilon)^{1-(1-\beta)\theta}\right)\right]^{1/\alpha}}{e(\omega^*(1+\epsilon)^{-(1-\beta)\theta})}
\end{aligned}
$$

- So when there is a shock, say $\epsilon \uparrow$, some agents do NOT fully adjust which means $\frac{M}{\overline{p}} \uparrow$ and $\omega^n \downarrow$. Actually the maximizers take into account the non-adjustment activity of the non-maximizers, so they don't adjust all the way to the optimal level without menu costs. Thus the price charged by the non-maximizers becomes relatively smaller than that of the maximizers relative to the general price level. So we get employment and output going up: $X \uparrow$, $N \uparrow \Longrightarrow$ Real Effects! Thus money is NOT neutral anymore.

- So the punchline of this model is that we can get nominal monetary shocks to matter by introducing price inertia or stickiness.

- Empirically, we see that both wages and prices are slow to adjust. The modal time between price changes for a retail item using newly available micro data from the BLS is about seven months. This is a fairly long time! For things like gasoline, firms can tell a cost story and say that if the price of oil goes up, the gas stations are forced to pass that on to their customers (immediately), however for most goods, firms are hurt by continually adjusting their prices. People don't like to see volatile toothpaste prices.

- **Remark** The interaction between the adjustors and non-adjustors is crucial. The two groups could either compliment each other or lead to substitutability where the more non-maximizers there are in the market, the less I (a maximizer) will want to adjust.

- **Remark** Note that a crucial component of having near rational agents is for markets to be imperfectly competitive. Imagine taking the downward sloping demand curve faced by the monopolist and increasing the elasticity. The cost of not adjusting goes to infinity as demand flattens out. Clearly with a PCM, charging a price above the market (optimal) price will lead to zero sales so the costs of not adjusting are huge.

23.2 Diamond/Howitt Trading/Transaction Cost Externality Model

- Consider an economy where it is costly to trade. Just as with search models of labor where frictions made it costly to search for a job, it is now costly to purchase items on the product market.
- Consider an island with coconut trees. Some of the trees are shorter so the cost of attaining the coconuts is relatively small compared to a tall tree. There is a taboo on the island that even though the only product is coconuts, you don't eat the coconuts you "produce", but instead want to trade for other's coconuts.
- Assume the return from production is as follows:

$$R = yp(Y),$$

  where y is your own production and $p(Y)$ is the price of coconuts as a function of aggregate coconut production.
- Two key assumptions:
  - (1) $p(Y) < 1$, ie, it is costly to sell coconuts (can't trade one for one – maybe you lose some coconut milk in the trade).
  - (2) $p'(Y) > 0$, ie, the frictions involved with selling coconuts get smaller when a lot of coconuts are being traded.
- Thus when the market is thicker, the transaction costs are lower.
- We will see that in these types of models, we will get multiple rational expectations equilibria. If we all conjecture that now is a good time to trade, it WILL be a good time to trade. If we all think no one else is going to trade, then it is in everyone's interest NOT to trade. We will seek some sort of Nash equilibrium but it will turn out that it is not unique.

24 Lecture 24: April 27, 2005

24.1 More on the Diamond/Howitt Externality Model

- Recall our model from last time:

$$R = yp(Y), \;\; p(Y) < 1, \;\; p'(Y) > 0.$$

- Suppose the agent faces costs $\alpha_t b_{it}$ where α_t is a common cost and b_{it} is idiosyncratic that follows distribution $h(\cdot)$. Thus, produce if:

$$yp(Y) \geq \alpha_t b_{it} \Longrightarrow b_{it} \leq \frac{yp(Y)}{\alpha_t}.$$

- Thus, total output in the economy can be written:

$$Y_t = y \int_0^{yp(Y)/\alpha_t} h(b)db.$$

- In equilibrium the following will hold:

$$F(Y_t) = Y_t - y \int_0^{yp(Y)/\alpha_t} h(b)db = 0.$$

- Thus if $F(Y_t)$ is monotonic, there will be only ONE solution since $F(\cdot)$ will only cross the axis once. See G-24.1. Consider differentiating F with respect to Y:

$$F'(Y_t) = 1 - yh(b^*)\frac{yp'(Y)}{\alpha_t} = 1 - \frac{y^2}{\alpha_t}p'(Y)h(b^*), \text{ with } b^* = yp(Y)/\alpha_t.$$

  If $p'(Y) < 0$, then $F'(Y) > 0$ and we get uniqueness. If $p'(Y) > 0$, we get multiple equilibria as shown in the graph. A necessary condition for mulitple equilibria is for $p'(Y) > 0$, or more generally, for there to be a POSITIVE externality.

- **Remark** Once again, given the three equilibria in G-24.1, which one we end up at is impossible to know. In fact, if agents all think that there will be low activity, it is in their best interest not to trade and hence, there WILL be low activity. So we have a self-fulfilling prophecy. Once we are at a point like A, there is also nothing in the model that can make us jump to a point like B. Hence we will introduce stability properties in the next model. Finally, this idea can also be extended to other things like: "Greenspan is so important to our economic stability only because we think he is."

- What do we think that $p(Y)$ looks like? See G-24.2. There is a region where $p(Y)$ is small so adding a bit of activity really doesn't help in reducing transaction costs, and then there is a region where a thicker market really makes a difference, and finally, it levels off as we get conjestion externalities with too many agents trying to trade.

24.2 Cooper and John - Externality Model

- Consider I agents, each of whom select an action, $e_i \in [0, E]$, yielding the general payoff:
$$\sigma(e_i, e_{-i}, \theta).$$
The action could be a pricing decision of a firm, the consumption decision of an agent, etc.

- We focus on Symmetric Nash Equilibria (SNE) so we consider the specific payoff function:
$$V(e_i, \overline{e}),$$
which induces a symmetric solution:
$$e_i^*(\overline{e}) = \overline{e}.$$
So the action of all agents will end up being the same. In this notation, $\overline{e}$ and e_{-i} both mean the actions of all other agents.

- Let the action of all agents in equilibrium just be denoted: "e". Then, at equilibrium:
$$\frac{\partial V(e,e)}{\partial e_{\{1\}}} = V_1(e,e) = 0.$$
So the set of all SNE is:
$$S = \{e \in [0, E] | V_1(e,e) = 0\}.$$

- To insure the existence of at least one equilibrium, make the following two assumptions:
$$lim_{e \to 0} V_1(e,e) > 0 \;\; and \;\; lim_{e \to E} V_1(e,e) < 0.$$
So $V_1(e,e)$ is decreasing in e (and indeed, equals 0 at least once), which implies $V_{11}(e,e) < 0$. Thus the value function itself is a nice concave function.

- So consider two cases:
  - Case (1): $V_{12}(e,e) < 0$. So the change in the marginal effect of your action from a change in everyone elses's action is negative. Thus, note $V_1 = V_1(e_i^*, \overline{e}) = 0$, so:
  $$\frac{\partial e_i^*}{\partial \overline{e}} = -\frac{V_{12}}{V_{11}} = \rho < 0.$$
  See G-24.3. Note this function slopes down and crosses the 45 only once so we have "Strategic Substitutability". Ie, the more other agents do an action, the less that I want to do. This setup gives us uniqueness: $e_i^* = \overline{e}^*$ at only one point.

– Case (2): $V_{12}(e,e) > 0$. So the change in the marginal effect of your action from a change in everyone elses's action is postive. Thus, note $V_1 = V_1(e_i^*, \overline{e}) = 0$, so:

$$\frac{\partial e_i^*}{\partial \overline{e}} = -\frac{V_{12}}{V_{11}} = \rho > 0.$$

See G-24.4. Note this function slopes up and could cross the 45 many times so we have "Strategic Complementarity". Ie, the more other agents do an action, the more that I want to do. This setup gives us a multiplicity of equilibria. This is just like the Diamond/Howitt model (the more trading that is happening, the cheaper it is to trade so I also want to trade).

- So a necessary condition for multiple SNE is for $\rho > 1$. Simply having it positive does not necessarily mean it crosses the 45 more than once.
- So we have the following results:

$$V_{12}(e_i^*, \overline{e}) < 0 \Longrightarrow \text{Strategic Substitutability}$$

$$V_{12}(e_i^*, \overline{e}) > 0 \Longrightarrow \text{Strategic Complementarity}$$
$$V_2(e_i^*, \overline{e}) > 0 \Longrightarrow \text{Positive Spillovers}$$
$$V_2(e_i^*, \overline{e}) < 0 \Longrightarrow \text{Negative Spillovers}$$

- Next we consider what a social planner would do (Cooperative equilibrium). The social planner maximizes:

$$Max_e \, V(e,e).$$

FOC and SOC:

$$V_1(e,e) + V_2(e,e) = 0.$$
$$V_{11}(e,e) + V_{12}(e,e) + V_{22}(e,e) + V_{21}(e,e) < 0.$$

Thus, the set of equilibria for the social planner is:

$$\tilde{S} = \{e \in [0,E] | V_1(e,e) + V_2(e,e) = 0, \; V_{11}(e,e) + 2 * V_{12}(e,e) + V_{22}(e,e) < 0\},$$

Noting that $V_{12}(e,e) = V_{21}(e,e)$. Generally $V_2 \neq 0$, so $S \neq \tilde{S}$.

- Thus anytime $V_2 \neq 0$ (spillovers exist), all the SNE we found above (G-24.4) will be inefficient from the social planner's point of view.
- **Remark** Given the FOC of the value function with respect to $\overline{e}$:

$$\underbrace{V_1(e_i(\overline{e}), \overline{e})}_{=0 \; at \; equilibrium} \frac{\partial e_i}{\partial \overline{e}} + V_2 = V_2,$$

this means that the SNE are "Pareto Ranked". Ie, in G-24.4, point S is worse than T is worse than U. Not sure where this point comes from.

- Next we consider multiplier effects which might be in this model. We say there is multiplier behavior if:
$$\left.\frac{\partial e_i^*}{\partial \theta}\right|_{e_{-i}\ constant} < \frac{de_i^*}{d\theta} < \sum_i \frac{de_i^*}{d\theta}.$$
So this says that if we have a shock to θ, then the individual effects holding other's actions constant is LESS than the effects when I take into account the actions of the other agents which is also LESS than the total action of all agents when they take into account the other's actions.

- Consider G-24.5. Here we assume $\rho \in (0,1)$. This means there could be multiple equilibria but the one we are at is stable. The shock shifts us up to a new curve and the effects are:
$$A = \frac{\partial e_i^*}{\partial \theta} << B = \frac{de_i^*}{d\theta}.$$
Since the equilibrium is symmetric, we can also assume,
$$B = \frac{de_i^*}{d\theta} << C = \sum \frac{de_i^*}{d\theta}.$$

- **Remark** As $\rho \to 1$, $\frac{de_i}{d\theta} \to \infty$. Thus we can make a small shock have arbitrarily large effects by changing ρ.

- Thus an open question might be: Is $\rho > 0$ and if so, how large is it? How big are the multiplier effects? If it is greater than 1, we would also get multiple equilibria. Empirically we estimate it to be pretty close to zero which means no multiple equilibria and very small multipliers.

- A final note on stability. See G-24.6. Given that all agent's update their expectations of other agent's actions according to:
$$E[\overline{e}] = \hat{\overline{e}} = e_{t-1},$$
we get the usual cobweb type diagram of stability. If $\rho \in (0,1)$, the equilibrium is stable and if $\rho > 1$, the equilibrium is unstable.

25 Lecture 25: May 2, 2005

25.1 More on Cooper & John

- The key results from the Cooper and John model were the possibility of multiple symmetric Nash equilibria and multiplier effects.
- Both of these things depended on the value of $\rho = -V_{12}/V_{11}$. If we assume V_{11} is negative, then $\rho > 0$ if V_{12} is positive, ie, if there are strategic complementarities. So when is $\rho > 0$? We offer three explanations:
  - (1) $\rho > 0$ if $\exists$ Trading Transaction cost Externalities (Diamond/Howitt). If trading volume makes trading easier (less costly) we might get complementarities. Empirically, Barsky and Warner try to test if transaction costs in, say, New York City are lower than in Iowa City. They find that the best time of the year to buy is christmas when there is a lot of buying and selling going on. However, in most cases, ρ is very difficult to estimate.
  - (2) $\rho > 0$ if $\exists$ Technological Externalities or External Increasing Returns. Consider the model:

    $$Max\, U(c_i, e_i), \text{ with } c_i = f(e_i, \bar{e}), \text{ and assume: } f_{12} > 0.$$

    So we have utility coming from consumption and effort which we can write in the Cooper/John notation as:

    $$Max_{e_i}\, V(e_i, \bar{e}) = U(f(e_i, \bar{e}), e_i) \Rightarrow V_1 = U_1 f_1 + U_2,\ V_{12} = U_{11} f_2 f_1 + U_1 f_{12} + \underbrace{U_{21} f_2}_{assume\ =0}.$$

    So our assumption that $f_{12} > 0$, is NOT enough for $V_{12} > 0$. Since $U_{11} < 0$, depending on the other terms, we might still get $V_{12} < 0$, (ie, $\rho < 0$). Anytime we have a model with concave utility, we get consumption smoothing which offsets the multiplier effects we would get for $\rho > 0$. Another extreme example is when $c_i = f(e_i, \bar{e}) = Min\{e_i, \bar{e}\}$. See G-25.1. Any point below e_0 is a SNE. So we can get multiple equilibria if there are such huge techological externalities. Empirically, Bartelsman, Cabalerro and Lyons use industry level data to estimate:

    $$y_{it} = A_{it} K_{it}^{\alpha} L_{it}^{\beta} Y_t^{\epsilon},$$

    and they find ϵ is much larger than zero which might induce multiple equilibria ($\rho > 1$). However, there are many econometric issues with estimating these parameters including utilization rates of the factors and the assumption of perfect competition. So others have found that $\epsilon \approx 0$.
  - (3) $\rho > 0$ if $\exists$ Demand Spillovers. Consider the case of Keynesian consumers and imperfect competition. Consumers consume a constant fraction of their income. So in a two sector model, the time line following a shock to sector 1 might be:

    $$Y_1 \downarrow \Rightarrow C_1 \downarrow \Rightarrow Y_2 \downarrow \Rightarrow C_2 \downarrow \Rightarrow Y_1 \downarrow \ldots.$$

This is the standard Keynesian multiplier effect and it could also give us $\rho > 0$.

- Note that explanations (1) and (3) rely on the idea that agents are **specialists in production and generalists in consumption.** Agents don't eat what they produce. If we have agents that smooth their consumption, explanations (2) and (3) are shot down.

- What about business cycles in the Cooper/John environment? How do we get fluctuations? Well, if $\rho > 0$ for any range, we can get movement as in G-25.2. We don't even require $\rho > 1$, but simply a positive value for ρ to get some (even small) multiplier effect which gives us real effects.

- As ρ increases, ie becomes more and more positive, both the shock and propagation mechanism are strong and longer. See G-25.3.

- **Remark** Some literature has focused on the idea of cycles as in G-25.4. We move from e_L to e_H depending on some (possibly external) cycle. The cause could be anything from monetary policy actions to sunspot activity. Whatever, the fact that people base their conjecture about what action everyone else is taking on something outside the model is perfectly rational and induces a self-fulfilling prophecy as we mentioned before. The real problem with this idea is still having $\rho > 1$ which induces the high and low stable equilibria to begin with.

- **Remark** Another idea would be to have nonlinearities in the best response function but they only really kick in when times are really bad. See G-25.5. ρ could even be positive over some range but we usually are working at a point like E and we need a large shock (to θ_1) to make the multiple equilibia kick in. 1930's idea. This idea seems more plausible than the last.

25.2 Unemployment

- Consider the labor market graph in G-25.6. Note we plot the nominal wage on the y-axis and the labor demand curve in nominal prices. A θ shock will push $p^0 MP_L$ back to $p^1 MP_L$ and the economy will move from point A. We could end up at C if everything works like it should and we have no real effects. Prices and wages fall by the same amount. If there are expectational errors, we might end up at a point like B because L^s didn't shift. If there is wage stickiness, we might even end up at B' since nominal wages are somehow fixed.

- Why are wages slow to adjust? Because it takes time to renegotiate wages. The labor market is NOT a spot market, always moving and adjusting to current conditions. Rather, contracts are made and when a wage is negotiated, the worker cares not only about his wage but also about his employment prospects over the life of the contract. So we will consider both an Implicit Contract Model and an Efficiency Wage model.

Implicit Contracts - Government Pays Unemployment Wage

- We assume that firms and workers are mobile in the long run but immobile in the short run.

- Consider a firm that maximizes expected profits:

$$Max_{w(p),L(p),m}\ E[\pi] = \int_{p=0}^{\infty} [p\ g(l(p)) - w(p)l(p)]f(p)dp,$$

  where labor demand, $l(p)$, and the wage paid, $w(p)$, are state dependent and the state of the world is determined by the price charged by the firm, p. $g[l(p)]$ is some sort of production function. m is the number of contracted workers.

- Reflecting the idea that workers care about both their wage and future employment prospects, we write the following Individual Rationality constraint:

$$IR: \quad \int_{p=0}^{\infty} \left[\frac{l(p)}{m}U(w(p)) + \Big(1 - \frac{l(p)}{m}\Big)U(Y)\right] f(p)dp \geq U(V).$$

  Where $l(p)/m$ is the probability that the worker is employed. If $l(p)$ is the total labor demand of the firm and m is the possible number of contracted workers, a fraction, $l(p)/m$, of the workers will be employed. Y is a monetary value of leisure and unemployment benefits paid by the government. V is the market determined contract value taken as given by the firm. $U(V)$ is the workers reservation utility.

- We also have the capacity constraint:

$$m \geq l(p).$$

- We make the following assumption: $V > Y$. So the market determined contract value must be more than the unemployment benefit or the workers would never work!

- Note that m is choosen by the firm before they know the realization of p. As we mentioned, labor demand and the wage are state contingent choice variables.

- More next time.

26 Lecture 26: May 4, 2004

26.1 More on Implicit Contracts

- We continue studying the implicit contract model where agents face unemployment income that is paid by the government. Firms have no control over this wage.

- Consider the firm's problem:

$$Max_{m,l(p),w(p)} \int_p [pg(l(p)) - w(p)l(p)]f(p)dp,$$

subject to:

$$m \geq l(p),$$

$$\int_p [\frac{l(p)}{m} U(w(p)) + (1 - \frac{l(p)}{m})U(Y)]f(p)dp \geq U(V).$$

- The lagrangian of this problem can be written:

$$\mathcal{L} = \int_p [pg(l(p)) - w(p)l(p)]f(p)dp + \lambda \left[\int_p [\frac{l(p)}{m} U(w(p)) + (1 - \frac{l(p)}{m})U(Y)]f(p)dp - U(V) \right] +$$

$$+ \int_p \mu(p)(m - l(p))f(p)dp.$$

- FOCs:

$$(l(p)): \quad pg'(l(p)) - w(p) + \lambda[\frac{U(w(p)) - U(Y)}{m}] - \mu(p) = 0. \quad (1)$$

$$(w(p)): \quad -l(p) + \lambda m^{-1} l(p) U'(w(p)) = 0. \quad (2)$$

$$(m): \quad \int_p [\lambda[-l(p)U(w(p))m^{-2} + l(p)U(Y)m^{-2}] + \mu(p)]f(p)dp = 0. \quad (3)$$

Note that m is choosen before firm's know the realization of p and hence the integral in the FOC. The other FOCs can be thought of as being satisfied for whatever is the state of the world, p. They must hold for ANY realization of p.

- So (2) implies:

$$1 = \lambda m^{-1} U'(w(p)).$$

$$\lambda = \frac{m}{U'(w(p))}. \quad (4)$$

Since λ and m are not state (p) dependent and $U'' < 0$, then this equation identifies a single equilibrium wage: $w(p) = \overline{w} \; \forall \; p$.

- Taking this result into (1):

$$pg'(l(p)) - \overline{w} + \lambda[\frac{U(\overline{w}) - U(Y)}{m}] - \mu(p) = 0.$$

$$pg'(l(p)) - \overline{w} + \frac{m}{U'(\overline{w})}[\frac{U(\overline{w}) - U(Y)}{m}] - \mu(p) = 0.$$

$$pg'(l(p)) = \overline{w} + \frac{U(Y) - U(\overline{w})}{U'(\overline{w})} + \mu(p). \quad (4)$$

- From (3):

$$\int_p \left[\lambda[-l(p)U(\overline{w})m^{-2} + l(p)U'(Y)m^{-2}] + \mu(p) \right] f(p)dp = 0.$$

$$\int_p \left[\frac{m}{U'(\overline{w})}[-l(p)U(\overline{w})m^{-2} + l(p)U(Y)m^{-2}] + \mu(p) \right] f(p)dp = 0.$$

$$\int_p \left[\frac{l(p)}{m}[\frac{U(Y) - U(\overline{w})}{U'(\overline{w})}] + \mu(p) \right] f(p)dp = 0.$$

$$\int_p \mu(p)f(p)dp = -\int_p \left[\frac{l(p)}{m}[\frac{U(Y) - U(\overline{w})}{U'(\overline{w})}] \right] f(p)dp. \quad (5)$$

- We also know the IR constraint will always bind so:

$$\int_p [\frac{l(p)}{m}U(w(p)) + (1 - \frac{l(p)}{m})U(Y)]f(p)dp = U(V).$$

$$\int_p [\frac{l(p)}{m}[U(\overline{w}) - U(Y)] + U(Y)]f(p)dp = U(V).$$

$$\int_p [\frac{l(p)}{m}[U(\overline{w}) - U(Y)]]f(p)dp = U(V) - U(Y).$$

$$\int_p [\frac{l(p)}{m}[\frac{U(Y) - U(\overline{w})}{U'(\overline{w})}]]f(p)dp = \frac{U(Y) - U(V)}{U'(\overline{w})}. \quad (6)$$

- Integrate (4) and plug in (5) and (6):

$$\int_p pg'(l(p))f(p)dp = \overline{w} + \frac{U(Y) - U(\overline{w})}{U'(\overline{w})} + \int_p \mu(p)f(p)dp.$$

$$\int_p pg'(l(p))f(p)dp = \overline{w} + \frac{U(Y) - U(\overline{w})}{U'(\overline{w})} - \int_p \left[\frac{l(p)}{m}[\frac{U(Y) - U(\overline{w})}{U'(\overline{w})}] \right] f(p)dp.$$

$$\int_p pg'(l(p))f(p)dp = \overline{w} + \frac{U(Y) - U(\overline{w})}{U'(\overline{w})} - \frac{U(Y) - U(V)}{U'(\overline{w})}.$$

$$\int_p pg'(l(p))f(p)dp = \overline{w} + \frac{U(V) - U(\overline{w})}{U'(\overline{w})}. \quad (8)$$

Finally.

- Thus firm's choose one level of m (the number of contracted workers) while they choose entire schedules for l and w. By the concavity of the utility function, equation (4) implies a fixed wage. Thus the optimal wage contract is a fixed wage! The labor market is not failing here, an optimal wage is actually the best strategy. The reason is that we have a risk neutral firm and a risk adverse worker so the optimal solution is to the transfer all the risk on to the firm.

- See G-26.1 We see that in periods of high demand, the firm is actually constrained by the capacity constraint and $m = l(p)$. They would like to hire more workers but they have to set labor equal to m since that's all they have contracted for. In periods of low demand, firms hire m_0 workers. However a social planner would simply try to set the value of worker's marginal product equal to their opportunity cost, Y. We have drawn Y above X in the diagram. We show graphically in G-26.2 that simply because the utility function is strictly concave, this relationship holds. We assumed $V > Y$ and it follows that $\overline{w} > V$ which means $\overline{w} > Y$. Thus we want to show:

$$Y > \overline{w} + \frac{U(Y) - U(\overline{w})}{U'(\overline{w})},$$

or in terms of our diagram:

$$B < \frac{A\overline{w}}{C}.$$

Since

$$U'(\overline{w}) = \frac{c}{\overline{w}} < \text{Slope of EF} = \frac{A}{B},$$

we have proven our claim.

- But what does this mean? It means the social planner would choose a point like A in G-26.1, resulting in employment $m_{sp} < m_0$. Thus the punchline of the model is that we have OVERemployment instead of underemployment. So we are not at an efficient outcome. The intuition is that the firm is trying to take all risk away from the worker. Since they don't control the unemployment wage, instead they try to increase employment above m_{sp} in order to reduce some of the worker's risk. They can't increase to FULL employment because it would be too costly, but in the end, too many workers are being hired.

- So what is the OPTIMAL contract? In the Rosen paper, he allows for the unemployment wage to be set by the firm (see homework). The firm's problem becomes:

$$Max_{m,l(p),w^e(p),w^u(p)} \int_p [pg(l(p)) - w^e(p)l(p) - w^u(m - l(p))]f(p)dp,$$

subject to:

$$m \geq l(p),$$

$$\int_p [\frac{l(p)}{m}U(w^e(p)) + (1 - \frac{l(p)}{m})U(Y + w^u(p))]f(p)dp \geq U(V).$$

- FOCs:

$$pg'(l(p)) = Y + \mu(p).$$

$$w^e(p) = V = w^u(p) + Y.$$

$$\int_p pg'(l(p))f(p)dp = V.$$

- So in this case we have COMPLETE risk shifting from the firm to the worker. Workers always get V and this coincides to the social planner's solution.
- This is actually an EQUILIBRIUM model! All markets clear, but we still get wage rigidity. This was a blow to the Keynesians who were looking for a market failure to explain sticky wages. The idea might be that in the long run, the labor market will clear, but in the short run we might get Keynesian effects.
- So this leads directly into other issues in labor markets including Moral Hazard and Adverse Selection. There are problems in motivating high effort and also in verifying the conditions on the contracts we just have been deriving. A firm might have an incentive to keep profits from the worker and possibly cheat the worker and the worker may have an incentive to shirk on the job.

26.2 Efficiency Wages - Shapiro/Stiglitz

- We make no assumptions about the risk aversion of the agents. Workers have utility:

$$U = w - e,$$

over income and effort. e can take on two values:

$$e = \{\bar{e}, 0\},$$

for high and no effort.

- Denote value functions:

$$V_E^N: \text{ Value of a Non-shirking Worker},$$

$$V_E^S: \text{ Value of a Shirking Worker}.$$

- Consider a production function of the form $g(eL)$, so since we have homogenous workers, if shirking is optimal, $e = 0$, and production will be zero. Thus we will seek a NO-shirking equilibrium.
- We'll show next time that $V_E^N \geq V_E^S$.

27 Lecture 26: May 4, 2004

27.1 More on Efficiency Wages

- Consider a utility function of the form $U = w - e$, where $e \in (0, \bar{e})$.
- Define the value functions of the shirkers and non-shirkers as:

$$rV_E^S = w + (b+q)(V_U - V_E^S),$$

$$rV_E^N = w - \bar{e} + b(V_U - V_E^N).$$

  So since we're in continuous time, we have the value equal to a current flow plus some expected capital gain or loss. w is the working wage, b is the probability of job destruction (exogeneous), and q is the probability that a shirker is caught and laid off. Clearly there is a tradeoff from the higher effort and the higher probability of getting caught (and fired).

- Since the production is of the form $g(eL)$ and we have homogenous workers, if one worker shirks, they all shirk and production is zero. Hence we will seek a non-shirking equilibrium where $e_i = \bar{e} \; \forall \; e_i$. Thus we need:

$$V_E^N \geq V_E^S.$$

  Denote the value of the non-shirkers (in equilibrium) as V^E.

- We also have unemployment value:

$$rV_U = \overline{w} + a(V^E - V_U),$$

  where a is the escape rate from unemployment and $\overline{w}$ is the unemployment wage.

- Thus, if we use the incentive compatibility condition above, we get:

$$w \geq \overline{w} + \bar{e} + \frac{\bar{e}}{q}(a + b + r).$$

  See paper for derivation but it gets messy.

- If we consider two definitions:

$$bL = a(N - L),$$

  or inflows into unemployment must equal outflows from unemployment in equilibrium. And:

$$u = \frac{N - L}{N},$$

  the unemployment rate. Then, our efficiency wage becomes:

$$\hat{w} = w \geq \overline{w} + \bar{e} + \frac{\bar{e}}{q}(\frac{b}{u} + r).$$

- See G-27.1 for a graph of the steady state. Note that the efficiency wage goes to infinity as unemployment goes to zero. Unemployment is the punishment for shirking so if there is no chance of being unemployed, all workers would shirk. Note there is also no reputation argument happening here: I don't hurt my chances of finding a new job just because I got caught slacking on my old job.

- **Remark** So in this model we have a market failure with less than efficienct employment (there are workers whose marginal product exceeds their opportunity cost that they are not hired and the unemployment is not voluntary). However, the unemployed cannot bid down the wage and get in the market because having a lower wage would result in shirking, zero production, and no employment.

- **Remark** This is a successful model of real wage rigidity but it is a static rigidity, not dynamic. If there is a shock to the economy and the marginal product curve shifts, wages move instantly. There is no slow adjustment. However, wages are still set above their market level so we have a static rigidity.

- To get nominal rigidity, we would have to introduce near rationality. Overall, this is a more realistic view of the labor market as it provides incentive problems resulting from asymmetric information. There are other versions of the model which use future wages to induce high effort. This might be more realistic. I'll work hard today for a normal wage in return for a much higher wage at some point in the future (if I don't shirk).

27.2 Empirical Work on the Labor Market

- So far, we have assumed homogenous agents (firms and workers), but in reality, there is quite a bit of movement of workers and firms.

- Define job creation and job destruction rates as:

$$JC = \sum_{e, g_{et} \geq 0} \frac{X_{et}}{X_t} g_{et},$$

$$JD = \sum_{e, g_{et} < 0} \frac{X_{et}}{X_t} |g_{et}|,$$

where,

$$g_{et} = \frac{E_{et} - E_{e,t-1}}{0.5(E_{et} + E_{e,t-1})} \approx log(E_{et}) - log(E_{e,t-1}),$$

and,

$$X_t = 0.5(E_t - E_{t-1}).$$

So the first term in the sum must be a weighting term of the firm relative to the sector and the second term is the growth rate of employment in the firm itself.

- The net job allocation rate is thus:

$$JC - JD = \sum_e \frac{X_{et}}{X_t} g_{et}.$$

- If JC and JD was about symmetric across time and states, we would be ok, but since there are frictions in the economy, churning of jobs becomes important. Frictions include:
  - (1) Cost of entry and exit of businesses.
  - (2) Labor/capital adjustment costs.
  - (3) Search/Matching frictions.
  - (4) Hetergeneity of job reallocation across time and place.
- In the data, we find that $JC > 10\%$ per year and $JD > 10\%$ per year so about 20% of jobs are reallocated in a given year. This is a lot of churning. Why does it happen?
- There are several properties of the Churning:
  - (1) Magnitude of the heterogeneity is large.
  - (2) The churning tends to be cyclical with $Var(JD) > Var(JC)$ at business cycle frequencies. See G-27.2. Following a recessions JD goes way up, but JC only falls a bit. It's much easier to fire someone than to hire.
  - (3) Concentrated. Much of JC and JD is accounted for by businesses for which $|g_{et}| > 20\%$. In the manufacturing sector, 2/3 of JC is by firms with $g_{et} > 20\%$ and 3/4 of JC is by firms with $|g_{et}| > 20\%$. Also, 20% of JC is from firm entry and 25% of JD is from firm exit. Thus firms open and close a lot.
  - (4) Job reallocation accounts for between 1/3 and 1/2 of worker reallocation. So workers reallocate more than jobs do. Separations and accessions total about 40% of jobs per year. So 40% of workers move jobs in a given year.
  - (5) Persistence: Businesses that engage in JC or JD do so over a long period of time.
  - (6) Productivity Enhancing. Virtually all job reallocation is productivity enhancing. We allocate resources to where they can be used most effectively. The US is very good at this and it's one of things driving its overall growth rate (even though reallocation is costly).

28 Lecture 28: May 11, 2004

28.1 Mortensen and Pissarides - Search Model

- This is the workhorse search model. It is actually an adjustment cost type model where we now know what the adjustment costs are instead of turning these costs into some sort of black box.
- The idea is there is a Friction involved with search and matching. You can view this two ways:
  - (1) Labor Market Perspective: There are search costs for the worker and vacancy creation costs for the firm.
  - (2) IO Perspective: There are entry and exit costs.
- We also assume there is some heterogeneity in the model in the form of an idiosyncratic productivity shock that is persistent. This shock has a time invariant distribution $F(\epsilon)$. Thus we can write productivity as:

  $$p + \sigma\epsilon,$$

  where p is a common productivity component, σ is a possibly time varying parameter, and ϵ is the shock.
- We assume that firms that enter a market have the LATEST technology. See see G-28.1. ϵ_u is the upper bound to productivity shocks and ϵ_d is the level of shock that would induce a firm to exit. Note that if there were NO frictions, an adverse productivity shock that pushed a firm/job below ϵ_u would cause them to exit immediately (close the job, close the firm, etc).
- But we DO have frictions so jobs that are between ϵ_d and ϵ_u will stay in the market and those below ϵ_d will leave.
- We assume that ϵ_d (JD) and the number of firms that enter the market (JC) are both endogenous (choice variables).
- **Punchline 1** $\epsilon_d \neq \epsilon_u$ if frictions exist, ie, if there are costs to reallocate resources.
- **Punchline 2** See G-28.2. Jobs in the segment, A, will get destroyed when we go from a boom to bust. This may account for the large spike in JD we see in the data following a recession. Also, as you go into a boom, ϵ_d shifts down and a bunch of new jobs are formed. Thus we get a lot of vacancies in booms.

The Model

- Matching Function:

  $$h = m(u, v),\ q = \frac{m}{v} < 1.$$

So hires are a function of unemployment and vacancies with $m_1 > 0$ and $m_2 > 0$. q is the probability that a vacancy is filled and because of frictions, it is less than one. This is hard to estimate but some have found that m exhibits CRS!

- Value of posting a vacancy to a firm:

$$rV = -c + q(\frac{v}{u})(J(\epsilon_u) - V).$$

So the value equals some initial cost of posting, c, plus the the probability of filling the vacancy, $q(v/u)$, a function, times the excess value of a filled job. Note that the filled job has value $J(\epsilon_u)$ as we assumed jobs come in with the latest and greatest technologies.

- Free entry condition:

$$rV = 0.$$

Jobs will enter a market just up until the point when the value of a new vacancy is zero.

- Surplus of a filled job:

$$S(\epsilon) = J(\epsilon) + W(\epsilon) - U,$$

So the surplus equals the value to the firm of the filled job plus the value to the worker of being in that filled job less his unemployment value.

- This surplus is going to be split (via some bargaining mechanism) between the worker (β) and the firm ($1 - \beta$):

$$\beta S(\epsilon) = W(\epsilon) - U,$$
$$(1 - \beta)S(\epsilon) = J(\epsilon).$$

- Value of a filled job to a firm:

$$rJ(\epsilon) = \underbrace{p + \sigma\epsilon}_{productivity} - \underbrace{w(\epsilon)}_{wage} + \underbrace{\lambda(1 - \beta)\int_{\epsilon_d}^{\epsilon_u}(S(x) - S(\epsilon))dF(x)}_{Option\ Value},$$

where this option value of getting hit by a shock is weighted by the probability of a shock, λ, and the proportion of the surplus that goes to the firm, $1-\beta$. Note the shock and resulting surplus could be positive or negative.

- Value of a filled job to a worker:

$$rW(\epsilon) = w(\epsilon) + \lambda\beta\int_{\epsilon_d}^{\epsilon_u}(S(x) - S(\epsilon))dF(x).$$

- Value of unemployment to the worker:

$$rU = b + \frac{Vq}{u}(W(\epsilon) - U),$$

where b is unemployment benefits and leisure, and $Vq/u = m/u = h/u \equiv$ probability of finding a new job.

- Value of a filled job at productivity level, ϵ_d:

$$J(\epsilon_d) = 0.$$

- **Remark** If $c = 0$, then $\epsilon_u = \epsilon_d$, because it is costless to post a vacancy.

- So a miracle occurs and we get to the steady state...

The Steady State

- Job Destruction Equation:

$$p + \sigma\epsilon_d = b + \frac{\beta c}{1-\beta}\frac{v}{u} - \frac{\sigma\lambda}{1-\lambda}\int_{\epsilon_d}^{\epsilon_u}(1 - F(x))dx.$$

So productivity at ϵ_d equals the opportunity cost of the job plus a positive term which reflects the possibility that workers can get another job (which increases ϵ_d), plus a negative term which reflects the chance that the firm is hit with a positive shock and things improve (which decreases ϵ_d).

- Imposing $rV = 0$ and doing integration by parts yields the Job Creation Equation:

$$q = \frac{c}{1-\beta}\frac{r+\lambda}{\sigma(\epsilon_u - \epsilon_d)}.$$

Model Extensions

- (1) Include On-the-job search.
- (2) Include the determination of $w(\epsilon)$ in the model.
- (3) Include a richer "c" term: make the frictions a function of the state of the economy. Maybe it is cheaper to create jobs in recessions because it is easier to find workers.
- (4) Include the role of Institutions in impacting the hiring/firing and entry/exit costs.

29 Final Review

29.1 Models

- Two types of shocks: nominal (monetary) and real (technology, cost, fiscal policy, structural changes, confidence).

Lucas Islands

- Supply curve:
$$y_t^i = b(p_t^i - E[p_t|I_t^i]).$$
- Signal Extraction:
$$p_t^i = p_t + z_t^i.$$
- Solution:
$$E[p_t|I_t^i] = (1-\theta)E[p_t|I_t] + \theta p_t^i,$$
$$\theta = \frac{\sigma_p^2}{\sigma_z^2 + \sigma_p^2}.$$
- Demand:
$$y_t = m_t - p_t.$$
- Supply = Demand implies:
$$E[m_t|I_t] = E[p_t|I_t].$$
- Solve for p and y:
$$p_t = \frac{1}{1+\beta}(\beta E[m_t|I_t] + m_t),$$
$$y_t = \frac{\beta}{1+\beta}(m_t - E[m_t|I_t]).$$
- **Punchline:** Only unanticipated money matters. No persistence from shocks.
- **Remark** Data show that nominal money shocks could matter. The reasons include 1) Creditibility, expectations formation and learning, 2) Price and wage stickiness, and 3) Credit versus money view of monetary policy.
- **Remark** With Lucas islands we have forward looking (rational) expectations so we don't get persistence. With adaptive (backward looking) expectations, we can get persistence. Amount of persistence depend on how quickly agent's expectations adjust.
- **Remark** Lucas's Challenge: Get serial correlation in output w/o serial correlation in the shocks.

Long and Plosser

- RBC model with technological shocks.
- Utility:
$$U = Max\ E[\sum_t \beta^t u(C_t, Z_t)].$$
Consumption and leisure.
- Technology:
$$Y_{t+1} = F(L_t, X_t; \lambda_{t+1}),$$
with L as labor inputs, X as the matrix of intermediate goods and lambda as tech shocks.
- Functional forms:
$$u(C_t, Z_t) = \theta_0 log(Z_t) + \sum_{i=1}^{N} X_{ijt},$$
$$Y_{i,t+1} = \lambda_{i,t+1} L_{it}^{b_i} \prod_{j=1}^{N} X_{ijt}^{a_{ij}}.$$
- Production of goods will be used either for consumption or for the production of other goods depending on how valuable they are.
- **Punchline:** Production takes time and goods produce goods YIELDS persistence.
- **Remark** When we look at the data we see that sectoral growth rates tend to move together. Why the comovements? Possibilities: 1) common shocks - not sector specific and 2) Sectoral Complementarities - i) Input/Output matrix or ii) Technological externalities (knowledge spillovers) or iii) Demand externalities (keynesian consumers). — Shea paper focused on 2i.

Blinder/Fisher - Production Smoothing, Buffer Stock Model

- We need a durable in the model to get persistence – introduce inventories.
- Demand:
$$p_t = v_t P_t D(X_t),$$
where v_t is an idiosyncratic shock, P is the general price level and X is sales.
- Cost schedule:
$$c_t = P_t c(Y_t),\ c' > 0, c'' > 0.$$
convex costs.

- Inventories:
$$N_{t+1} = N_t + Y_t - X_t.$$
$Y_t \neq X_t$ which allows production smoothing.

- Profits:
$$\pi_t = R(X_t, v_t) - c(Y_t) - \frac{B(N_{t+1})}{1+r},$$
where the B function is the cost of maintaining inventories.

- **Punchline:** We get persistence following shocks through the buffer stock. This answered Lucas' challenge since even a SINGLE one period nomial (monetary) shock will generate persistent output movements for many periods. We should see $Var(Y_t) < Var(X_t)$, as firms production smooth to avoid the high convex costs.

- **Remark** In reality the opposite happens. Could just be data problems as physical unit data is hard to come by, there are accounting problems, and we also see plant level heterogeneity.

  Remark Alternatives: Non-convex models: fixed costs of settting up plants or running shifts, so the cost schedule could be convex but with non-convex jumps as you open new plants or shifts. Cost schedule variations (from real shocks) itself could be driving the high variance in output. Or Stockout avoidance as firms really don't like to be caught in a situation where they have unfilled orders. If costs are low for a while, they may build up inventories to prepare for a period of high demand.

Adjustment Cost Models

- Suppose there is some cost to adjusting your factors of production - training new labor, etc. If we have convex adjustment costs, we get persistence in the factors. In the micro level data, we don't get smooth adjustment processes following a shock but rather very lump adjustments. All or nothing type of thing. The distribution of log changes in kurtotic. About 17% of firms have essential no change in labor demand at a quarterly frequency. Lots of inertial behavior at the micro level.

- In macro level data, the adjustment is fairly smooth.

- In a generalized (s,s) model by Caballero and Engel, we have a model where firms have some range in inaction and once things get really out of shape, firms adjust. The cross sectional distribution of firms is key and this is what makes aggregation difficult and it reconciles the difference between the micro and macro data results.

- In another adjustment cost model, hour worked are used as a buffer stock compared to hiring and firing.

Search Frictions

- Summary of reasons for persistence:
  - (1) Adaptive Expectations.
  - (2) Capital Accumulation.
  - (3) Inventories.
  - (4) Adjustment Costs.
  - (5) Production takes time and goods produce goods.
  - (6) And NOW: Employment search frictions.

- Model:

$$\Delta u_t = \phi_t - \pi_t u_{t-1},$$

  where ϕ_t is the inflow rate into unemployment and π_t is the escape rate. $\pi < 1$ due to search frictions. The adjustment process following a shock depends crucially on π_t, the measure of search frictions.

- **Punchline:** This model yields persistence because it takes time for people to find jobs.

New Keynesian Models - Nominal Rigidities

- Now we turn to a setting where markets are NOT perfect and even nominal shocks may generate persistence through market failures and multiplier effects.

- Consider the graph of the envelope theorem ... The cost of small mistakes/inertial behavior is only second order in the shock while the social loss is first order and significant.

Akerlov and Yellen - Near Rational Agents

- Sales /Demand:

$$X = (\frac{p}{\overline{p}})^{-\eta}(\frac{M}{\overline{p}}).$$

- Production function:

$$X = (eN)^{\alpha}.$$

- Firm's problem:

$$Max_{p,\omega}\ p * (\frac{p}{\overline{p}})^{-\eta}(\frac{M}{\overline{p}}) - \omega\overline{p}N.$$

- Now β of the firms adjust to shocks and $1-\beta$ of the agents face menu costs so they are "near rational".

- **Punchline:** So the key result is that both output/sales and the demand for labor are dependent on nominal money shocks since some agents don't fully adjust.

- **Remark** The interaction between the adjustors and non-adjustors is crucial. The two groups could complement each other or lead to substitutability.
- **Remark** A key assumption of these models is imperfect competition. As demand becomes more elastic, the costs of not-adjusting become more and more severe, so no one adjusts.

Diamond/Howitt Trading/Transaction Costs Externalities

- Recall our model from last time:

$$R = yp(Y), \;\; p(Y) < 1, \;\; p'(Y) > 0.$$

- Costs $\alpha_t b_{it}$ where α_t is a common cost and b_{it} is idiosyncratic with distribution $h(\cdot)$. Thus, produce if:

$$yp(Y) \geq \alpha_t b_{it} \Longrightarrow b_{it} \leq \frac{yp(Y)}{\alpha_t}.$$

- Thus, total output:

$$Y_t = y \int_0^{yp(Y)/\alpha_t} h(b)db.$$

- In equilibrium:

$$F(Y_t) = Y_t - y \int_0^{yp(Y)/\alpha_t} h(b)db = 0.$$

- Thus if $F(Y_t)$ is monotonic, there will be only ONE solution.
- If $p'(Y) < 0$, then $F'(Y) > 0$ and we get uniqueness. If $p'(Y) > 0$, we get multiple equilibria as shown in the graph. A necessary condition for mulitple equilibria is for $p'(Y) > 0$, or more generally, for there to be a POSITIVE externality.
- **Punchline:** Equilibria can be self-fulfilling.
- **Remark** "Greenspan is so important to our economic stability only because we think he is."

Cooper and John - Externality Model

- Agent's choose an action, $e_i \in [0, E]$.
- We focus on Symmetric Nash Equilibria (SNE) so we consider the specific payoff function:

$$V(e_i, \bar{e}),$$

which induces a symmetric solution:

$$e_i^*(\bar{e}) = \bar{e}.$$

- So the set of all SNE is:

$$S = \{e \in [0, E] | V_1(e, e) = 0\}.$$

- Assume $V_1(e, e)$ is decreasing.
- Case (1): $V_{12}(e, e) < 0$.

$$\frac{\partial e_i^*}{\partial \overline{e}} = -\frac{V_{12}}{V_{11}} = \rho < 0.$$

  'Strategic Substitutability".
- Case (2): $V_{12}(e, e) > 0$.

$$\frac{\partial e_i^*}{\partial \overline{e}} = -\frac{V_{12}}{V_{11}} = \rho > 0.$$

  'Strategic Complementarity".
- So a necessary condition for multiple SNE is for $\rho > 1$. Simply having it positive does not necessarily mean it crosses the 45 more than once.
- So we have the following results:

$$V_{12}(e_i^*, \overline{e}) < 0 \Longrightarrow \text{Strategic Substitutability}$$

$$V_{12}(e_i^*, \overline{e}) > 0 \Longrightarrow \text{Strategic Complementarity}$$

$$V_2(e_i^*, \overline{e}) > 0 \Longrightarrow \text{Positive Spillovers}$$

$$V_2(e_i^*, \overline{e}) < 0 \Longrightarrow \text{Negative Spillovers}$$

- Multiplier effects if:

$$\left.\frac{\partial e_i^*}{\partial \theta}\right|_{e_{-i}\ constant} < \frac{de_i^*}{d\theta} < \sum_i \frac{de_i^*}{d\theta}.$$

- **Punchline** The key results from the Cooper and John model were the possibility of multiple symmetric Nash equilibria and multiplier effects.
- **Remark** So when is $\rho > 0$? We offer three explanations:
  - (1) $\rho > 0$ if $\exists$ Trading Transaction cost Externalities (Diamond/Howitt).
  - (2) $\rho > 0$ if $\exists$ Technological Externalities or External Increasing Returns. Anytime we have a model with concave utility, we get consumption smoothing which offsets the multiplier effects we would get for $\rho > 0$. Another extreme example is when $c_i = f(e_i, \overline{e}) = Min\{e_i, \overline{e}\}$.
  - (3) $\rho > 0$ if $\exists$ Demand Spillovers. Consider the case of Keynesian consumers and imperfect competition.
- **Remark** Agents are specialists in production and generalists in consumption.

- **Remark** What about business cycles in the Cooper/John environment? How do we get fluctuations? Well, if $\rho > 0$ for any range, we can get movement as in G-25.2. We don't even require $\rho > 1$, but simply a positive value for ρ to get some (even small) multiplier effect which gives us real effects.

- **Remark** Another idea is nonlinearities in the best response function but they only really kick in when times are really bad.

Unemployment Models

- Why is the labor market (wages) slow to adjust? It takes time to renegotiate wages. It's costly to adjust your wage every day. Hence the need for contracts.

Implicit Contracts - Government Pays Unemployment Wage

- We assume that firms and workers are mobile in the long run but immobile in the short run.

- Consider a firm that maximizes expected profits:

$$Max_{w(p),L(p),m}\ E[\pi] = \int_{p=0}^{\infty} [p\ g(l(p)) - w(p)l(p)] f(p)dp,$$

- Individual Rationality constraint:

$$IR: \quad \int_{p=0}^{\infty} \left[\frac{l(p)}{m} U(w(p)) + \left(1 - \frac{l(p)}{m}\right) U(Y) \right] f(p)dp \geq U(V).$$

- We also have the capacity constraint:

$$m \geq l(p).$$

- Result: single equilibrium wage: $w(p) = \overline{w}\ \forall\ p$.

- See G-26.1 We see that in periods of high demand, the firm is actually constrained by the capacity constraint and $m = l(p)$. They would like to hire more workers but they have to set labor equal to m since that's all they have contracted for. In periods of low demand, firms hire m_0 workers. However a social planner would simply try to set the value of worker's marginal product equal to their opportunity cost, Y. We have drawn Y above X in the diagram. We show graphically in G-26.2 that simply because the utility function is strictly concave, this relationship holds. We assumed $V > Y$ and it follows that $\overline{w} > V$ which means $\overline{w} > Y$.

- But what does this mean? It means the social planner would choose a point like A in G-26.1, resulting in employment $m_{sp} < m_0$.

- **Punchline** So we have OVERemployment instead of underemployment. So we are not at an efficient outcome. The intuition is that the firm is trying to take all risk away from the worker. Since they don't control the unemployment wage, instead they try to increase employment above m_{sp} in order to reduce some of the worker's risk. They can't increase to FULL employment because it would be too costly, but in the end, too many workers are being hired.

Implicit Contracts - Rosen: Firm Pays Unemployment Wage

- The firm's problem becomes:

$$Max_{m,l(p),w^e(p),w^u(p)} \int_p [pg(l(p)) - w^e(p)l(p) - w^u(m - l(p))]f(p)dp,$$

subject to:

$$m \geq l(p),$$
$$\int_p [\frac{l(p)}{m}U(w^e(p)) + (1 - \frac{l(p)}{m})U(Y + w^u(p))]f(p)dp \geq U(V).$$

- FOCs:

$$pg'(l(p)) = Y + \mu(p).$$
$$w^e(p) = V = w^u(p) + Y.$$
$$\int_p pg'(l(p))f(p)dp = V.$$

- So in this case we have COMPLETE risk shifting from the firm to the worker. Workers always get V and this coincides to the social planner's solution.
- This is actually an EQUILIBRIUM model! All markets clear, but we still get wage rigidity. This was a blow to the Keynesians who were looking for a market failure to explain sticky wages. The idea might be that in the long run, the labor market will clear but in the short run we might get Keynesian effects.

Efficiency Wages - Shapiro/Stiglitz

- We make no assumptions about the risk aversion of the agents. Workers have utility:

$$U = w - e, \; e = \{\bar{e}, 0\}.$$

- Production function of the form $g(eL)$, so since we have homogenous workers, if shirking is optimal, $e = 0$, and production will be zero. Thus we will seek a NO-shirking equilibrium.

- Define the value functions of the shirkers and non-shirkers as:

$$rV_E^S = w + (b+q)(V_U - V_E^S),$$

$$rV_E^N = w - \overline{e} + b(V_U - V_E^N).$$

- We also have unemployment value:

$$rV_U = \overline{w} + a(V^E - V_U),$$

- Thus, if we use the incentive compatibility condition above, we get:

$$w \geq \overline{w} + \overline{e} + \frac{\overline{e}}{q}(a+b+r).$$

- If we consider two definitions:

$$bL = a(N - L),$$

$$u = \frac{N-L}{N},$$

  Then, our efficiency wage becomes:

$$\hat{w} = w \geq \overline{w} + \overline{e} + \frac{\overline{e}}{q}(\frac{b}{u} + r).$$

- **Remark** Unemployment is the punishment for shirking so if there is no chance of being unemployed, all workers would shirk.

- **Remark** So in this model we have a market failure with less than efficiency employment (there are workers whose marginal product exceeds their opportunity cost and they are not hired, and the unemployment is not voluntary. However, the unemployed cannot bid down the wage and get in the market because having a lower wage would result in shirking, zero production, and no employment.

- **Remark** This is a successful model of real wage rigidity but it is a static rigidity, not dynamic. If there is a shock to the economy and the marginal product curve shifts, wages move instantly. There is no slow adjustment. However, wages are still set above their market level so we have a static rigidity.

Empirical Work on the Labor Market

- There are frictions in the labor market: (how does this relate) ?
  - (1) Cost of entry and exit of businesses.
  - (2) Labor/capital adjustment costs.
  - (3) Search/Matching frictions.

- In the data, we find that $JC > 10\%$ per year and $JD > 10\%$ per year so about 20% of jobs are reallocated in a given year. This is a lot of churning.
- There are several properties of the Churning:
  - (1) Magnitude of the heterogeneity is large.
  - (2) The churning tends to be cyclical with $Var(JD) > Var(JC)$ at business cycle frequencies. See G-27.2. Following a recessions JD goes way up, but JC only falls a bit. It's much easier to fire someone than to hire.
  - (3) Concentrated. Much of JC and JD is accounted for by businesses for which $|g_{et}| > 20\%$. In the manufacturing sector, 2/3 of JC is by firms with $g_{et} > 20\%$ and 3/4 of JC is by firms with $|g_{et}| > 20\%$. Also, 20% of JC is from firm entry and 25% of JD is from firm exit. Thus firms open and close a lot.
  - (4) Job reallocation accounts for between 1/3 and 1/2 of worker reallocation. So workers more more than jobs do. Separations and accessions total about 40% of jobs per year. So 40% of workers move jobs in a given year (though there may be double counting).
  - (5) Persistence: Businesses that engage in JC or JD do so over a long period of time.
  - (6) Productivity Enhancing. Virtually all job reallocation is productivity enhancing. We allocate resources to where they can be used most effectively. The US is very good at this and is one of things driving its overall growth rate (even though reallocation is costly).

Mortensen and Pissarides - Search Model

- The idea is there is a Friction involved with search and matching.
- We also assume there is some heterogeneity in the model in the form of an idiosyncratic productivity shock that is persistent.
- We assume that ϵ_d (JD) and the number of firms that enter the market (JC) are both endogenous (choice variables).
- **Punchline 1** $\epsilon_d \neq \epsilon_u$ if frictions exist, ie, if there are costs to reallocate resources.
- **Punchline 2** See G-28.2. Jobs in the segment, A, will get destroyed when we go from a boom to bust. This may account for the large spike in JD we see in the data following a recession. Also, as you go into a boom, ϵ_d shifts down and a bunch of new jobs are formed. Thus we get a lot of vacancies in booms.
- Matching Function:
$$h = m(u, v),\ q = \frac{m}{v} < 1.$$

- Value of posting a vacancy to a firm:

$$rV = -c + q(\frac{v}{u})(J(\epsilon_u) - V).$$

- Free entry condition:

$$rV = 0.$$

- Surplus of a filled job:

$$S(\epsilon) = J(\epsilon) + W(\epsilon) - U,$$
$$\beta S(\epsilon) = W(\epsilon) - U,$$
$$(1-\beta)S(\epsilon) = J(\epsilon).$$

- Value of a filled job to a firm:

$$rJ(\epsilon) = \underbrace{p + \sigma\epsilon}_{productivity} - \underbrace{w(\epsilon)}_{wage} + \underbrace{\lambda(1-\beta)\int_{\epsilon_d}^{\epsilon_u}(S(x) - S(\epsilon))dF(x),}_{Option\ Value}$$

- Value of a filled job to a worker:

$$rW(\epsilon) = w(\epsilon) + \lambda\beta\int_{\epsilon_d}^{\epsilon_u}(S(x) - S(\epsilon))dF(x).$$

- Value of unemployment to the worker:

$$rU = b + \frac{Vq}{u}(W(\epsilon) - U),$$

- Value of a filled job at productivity level, ϵ_d:

$$J(\epsilon_d) = 0.$$

- **Remark** If $c = 0$, then $\epsilon_u = \epsilon_d$, because it is costless to post a vacancy.
- The Steady State:

$$JD: \quad p + \sigma\epsilon_d = b + \frac{\beta c}{1-\beta}\frac{v}{u} - \frac{\sigma\lambda}{1-\lambda}\int_{\epsilon_d}^{\epsilon_u}(1 - F(x))dx.$$

$$JC: \quad q = \frac{c}{1-\beta}\frac{r+\lambda}{\sigma(\epsilon_u - \epsilon_d)}.$$

- Model Extensions
  - (1) Include On-the-job search.
  - (2) Include the determination of $w(\epsilon)$ in the model.

- (3) Include a richer "c" term: make the frictions a function of the state of the economy. Maybe it is cheaper to create jobs in recessions because it is easier to find workers.
- (4) Include the role of Institutions in impacting the hiring/firing and entry/exit costs.

29.2 Problem Set Notes

- IS/LM. Market for goods and services (IS):

$$y = c(r, y^d) + I(r, y) + G, \quad y^d = (1-t)y.$$

Money market:

$$\frac{M}{p} = L(r, y).$$

Production:

$$y = F(N, K).$$

Wage Determination:

$$\frac{w}{p} = F_N(N, K).$$

Total differential:

$$dy = c_r dr + c_y[(1-t)dy - ydt] + I_r dr + I_y dy + dG.$$

$$\frac{dM}{p} - \frac{Mdp}{p^2} = L_r dr + L_y dy.$$

$$dy = F_N dN + F_K dK.$$

$$\frac{dw}{p} - \frac{wdp}{p^2} = F_{NN} dN + F_{NK} dK.$$

- Lucas Islands – Regardless of the monetary policy adopted, rational expectations imply that anticipated money does NOT matter for output (though it will affect prices). No persistence from shocks.

- Long and Plosser – RBC model with technological shocks. **Punchline:** Production takes time and goods produce goods YIELDS persistence.

- Blinder/Fisher - Production Smoothing, Buffer Stock Model: We need a durable in the model to get persistence – introduce inventories. We get persistence following shocks through the buffer stock. This answered Lucas' challenge since even a SINGLE one period nomial (monetary) shock will generate persistent output movements for many periods. We should see $Var(Y_t) < Var(X_t)$, as firms production smooth to avoid the high convex costs.

- Akerlof and Yellen - Menu costs and near rationality.

$$L(\epsilon) \approx L(\epsilon)\Bigg|_{\epsilon=0} + L'(\epsilon)\Bigg|_{\epsilon=0} \epsilon + \frac{1}{2}L''(\epsilon)\Bigg|_{\epsilon=0} \epsilon^2.$$

- Cooper and John - Externality Model. **Punchline** The key results from the Cooper and John model were the possibility of multiple symmetric Nash equilibria and multiplier effects.**Remark** Agents are specialists in production and generalists in consumption.

- Implicit Contracts - Government Pays Unemployment Wage. We assume that firms and workers are mobile in the long run but immobile in the short run. Result: single equilibrium wage: $w(p) = \overline{w} \ \forall \ p$. **Punchline** So we have OVERemployment instead of underemployment. So we are not at an efficient outcome. The intuition is that the firm is trying to take all risk away from the worker.

- Implicit Contracts - Rosen: Firm Pays Unemployment Wage. So in this case we have COMPLETE risk shifting from the firm to the worker. Workers always get V and this coincides to the social planner's solution. This is actually an EQUILIBRIUM model! All markets clear, but we still get wage rigidity. This was a blow to the Keynesians who were looking for a market failure to explain sticky wages. The idea might be that in the long run, the labor market will clear but in the short run we might get Keynesian effects.

- Recursive projection. Suppose your model is:

$$y = a_0 + a_1 x_1 + a_2 x_2 + \epsilon.$$

Suppose you initially only have data on x_1. Project y on a constant and x_1:

$$P(y|1, x_1).$$

Now we obtain data on x_2. Project x_2 on a constant and x_1:

$$P(x_2|1, x_1).$$

Now for the projection of y on a constant, x_1 and x_2:

$$P(y|1, x_1, x_2) = P(y|1, x_1) + P\left[\underbrace{y - P(y|1, x_1)}_{forecast\ error}\Bigg|\underbrace{x_2 - P(x_2|1, x_1)}_{forecast\ error}\right].$$