Economics 603: Microeconomics
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1 Lecture 1: August 31, 2004

1.1 Preferences

- Define the set of possible consumption bundles (an $nx1$ vector) as X. X is the "set of alternatives."
- Usually all elements of X should be non-negative, X should be closed and convex.
- Define the following relations:

$$\succ\text{: Strictly Preferred,}$$

$$\succeq\text{: Weakly Preferred,}$$

$$\sim\text{: Indifferent.}$$

- If $x \succ y$ then $x \succeq y$, $y \not\succeq x$.
- If $x \sim y$ then $x \succeq y$, $y \succeq x$.
- Usually we assume a few things in problems involving preferences.

Rational Assumptions

- Completeness: A consumer can rank any 2 consumption bundles, $x \succeq y$ and/or $y \succeq x$.
- Transitivity: If $x \succeq y$, $y \succeq z$, then $x \succeq z$. The lack of this property leads to a money pump.

Continuity Assumption

- $\succeq$ is continuous if it is preserved under limits. Suppose:

$$\{y_i\}_{i=1}^{n} \to y \; and \; \{x_i\}_{i=1}^{n} \to x.$$

  If for all i, $x_i \succeq y_i$, then $x \succeq y$ and $\succeq$ is continuous.

- The continuity assumption is violated with lexicographic preferences where one good matters much more than the other. Suppose good 1 matters more than good 2 such that you would only consider the relative quantities of good 2 if the quantity of good 1 was the same in both bundles. For example:

$$x_1^n = 1 + \frac{1}{n}, \;\; y_1^n = 1.$$

$$x_2^n = 0 \;\; y_2^n = 100.$$

  Then,

$$n = 1 \Longrightarrow (2, 0) \succeq (1, 100),$$

$$n = 2 \Longrightarrow (1.5, 0) \succeq (1, 100),$$
$$n = 3 \Longrightarrow (1.33, 0) \succeq (1, 100),$$
$$\vdots$$
$$limit \Longrightarrow (1, 0) \prec (1, 100).$$

So we lost continuity in the limit.

Desirability Assumptions

- $\succeq$ is Strongly Monotone if:
$$y \geq x, y \neq x \Rightarrow y \succ x.$$
- $\succeq$ is Monotone if:
$$y >> x, \Rightarrow y \succ x.$$
So strongly monotone is when at least one element of y is greater than x leads to preferring y over x. So in the 2 good case, both goods must matter to the consumer. If you increase one holding the other constant, if your preferences are strongly monotone, you MUST prefer this new bundle. With monotone, you only have to prefer a bundle y over a bundle x if EVERY element in y is greater than x. In the 2 good case, increasing the quantity of one good while leaving the other same may or may not leave the consumer indifferent between the two bundles. See graph in notes. [G-1.1].
- $\succeq$ exhibits local non-satiation if $\forall\, x \in X$ and $\epsilon > 0$,
$$\exists\, y \in X \ \ni \ ||y - x|| < \epsilon \textit{ and } y \succ x.$$
See graph in notes [G-1.2].
- Thus Strong Monotonicity $\Longrightarrow$ Monotonicity $\Longrightarrow$ Locally Non-Satiated Preferences.

Convexity Assumption

- $\succeq$ is strictly convex if:
$$y \succeq x, z \succeq x \textit{ and } y \neq z \Longrightarrow \alpha y + (1 - \alpha) z \succ x.$$
- $\succeq$ is convex if:
$$y \succeq x, z \succeq x \textit{ and } y \neq z \Longrightarrow \alpha y + (1 - \alpha) z \succeq x.$$
See graph in notes [G-1.3]
- Of course if preferences are strictly convex, they are also convex.

Proposition 3.c.1

- (MWG pg 47). If $\succeq$ is rational, continuous, and monotone, then there exists $u(\cdot)$ that represents $\succeq$.

Pf: Let $e = (1, \ldots, 1)$. Define for any $x \in X$,

$$u(x) = min\{\alpha \geq 0 : ae \succeq x\}.$$

Observe that the set in the definition of $u(\cdot)$ is nonempty, since by monotonicity, we can choose $\alpha > max\{x_1, \ldots, x_L\}$. By continuity, the minimum is attained and has the property $\alpha e \sim x$. We conclude that $u(\cdot)$ can be used as a utility funtion that represents $\succeq$. QED.

- See graph [G-1.4] in notes. e is just the unit vector. For any given bundle, we can take a multiple of e to get to something that we are indifferent between the bundle and x. Suppose $x = (6, 3)$ and we find that $(6, 3) \sim (4, 4)$ then $u(x) = 4$. So the utility function can map any bundle into a number so we have created a way to move from something real, like a person's preferences, to something more abstract, like a utility function.

2 Lecture 2: September 2, 2004

2.1 Upper Contour Sets and Indifference Curves

- Upper Contour Set:
$$\{x \in X : x \succeq y\}.$$
- Lower Contour Set:
$$\{x \in X : y \succeq x\}.$$
- Indifference Curve:
$$\{x \in X : x \sim y\}.$$
Thus the indifference curve is the intersection of the Upper Contour Set (UCS) and the Lower Contour Set (LCS). See graph in notes [G-2.1].
- Equivalent Definitions.
$$\text{Strictly Convex Preferences} \Longrightarrow \text{Upper Contour Set is Strictly Convex.}$$
$$\text{Convex Preferences} \Longrightarrow \text{Upper Contour Set is Convex.}$$
Continuous Preference Relation $\Longrightarrow$ UCS and LCS are Closed (contain their boundries.)
- See graph in notes for convex preferences which are not strictly convex [G-2.2].
- See graph of lexicographic preferences. With these preferences, the UCS and LCS are not closed and thus preferences do not satisfy continuity. [G-2.3].
- Utility functions. Key properties.
  - Concave if $u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y)$, $\forall\ \alpha \in [0,1]$.
  - Strictly Concave if $u(\alpha x + (1-\alpha)y) > \alpha u(x) + (1-\alpha)u(y)$, $\forall\ x \neq y, \alpha \in (0,1)$.
  - Quasi-Concave if $u(\alpha x + (1-\alpha)y) \geq min\{u(x), u(y)\}$, $\forall\ \alpha \in [0,1]$.
  - Strictly Quasi-Concave if $u(\alpha x + (1-\alpha)y) > min\{u(x), u(y)\}$, $\forall\ x \neq y, \alpha \in (0,1)$.
- See graphs in notes for concavity and quasi-concavity pictures. Note that concavity is not what we are interested in. The more important property for utility functions is quasi-concavity. [G-2.4].
- To check for quasi-concavity, select and x and y such that $u(x) = u(y)$ and then consider the convex combination.
- **Definition:** A utility function, $u(\cdot)$, is said to represent the preference relaton, $\succeq$, if for all $x, y \in X$,
$$x \succeq y \Longrightarrow u(x) \geq u(y).$$

- **Proposition:** Invariance of Monotonic Transformations. Suppose $u(\cdot)$ represents $\succeq$. Let $v(x) = f(u(x))$ where $f(\cdot)$ is any strictly increasing function on the real numbers. Then $v(\cdot)$ is also a utility function representing $\succeq$.

- See graph in notes which show that this is why concavity is not the key property for utility functions. [G-2.5]. If we start with some concave functions, we can take monotonic transformations and get convex functions, etc. Thus, the key property is that monotonic transformations of utility functions must not change their ordinal ranking. Monotonicity must be preserved.

- **Proposition:** Suppose $u(\cdot)$ represents $\succeq$. Then:
  - 1) $u(\cdot)$ is strongly increasing $\Longrightarrow \succeq$ is strongly monotonic.
  - 2) $u(\cdot)$ is quasi-concave $\Longrightarrow \succeq$ is convex.
  - 2) $u(\cdot)$ is strictly quasi-concave $\Longrightarrow \succeq$ is strictly convex.

2.2 The Consumer's Maximization Problem

- Define the following. Commodity Vector:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_L \end{pmatrix} \in X \subset \Re^L_+.$$

  So we have L commodities, X is closed, convex and the elements of X are non-negative.

- Price Vector:

$$p = \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_L \end{pmatrix} \in \Re^L_{++}.$$

  So $p_l > 0 \; \forall \; l = 1, ..., L$. Also assume p is determined exogeneously.

- Now suppose wealth $\equiv w > 0$. A scalar.

- Define the Walrasian/Competition Budget Set as:

$$B_{p,w} = \{x \in \Re^L_+ : px \leq w\}.$$

  See graph in notes [G-2.6]

- The consumer's problem is to find the optimal demand correspondence:

$$x(p, w) = \{x \in B_{p,w} : x \succeq y \; \forall \; y \in B_{p,w}\}.$$

- **Proposition:** (MWG 50-52, 3.d.1-2). If $\succeq$ is a rational, continuous, and locally non-satiated preference relation, then the consumers maximization problem has a solution and exhibits:
  - 1) Homogeneous of degree 0:
  $$x(\alpha p, \alpha w) = x(p, w) \; \forall \; p \in \Re^L_{++}, w > 0, \alpha > 0.$$
  - 2) Walras Law (Binding budget constraint).
  $$w = px \; \forall \; x \in x(p, w).$$
  - 3) If $\succeq$ is convex, then $x(p, w)$ is a convex set.
  - 4) If $\succeq$ is strictly convex, then $x(p, w)$ is single valued, or we call it a demand function.

- See graphs in notes [G-2.7] which show points 3 and 4. Proof: Any rational and continuous preference relation can be represented by some continuous utility function, $u(\cdot)$. Since $p_l > 0$ for all $l = 1, ..., L$, the Walrasian budget set is bounded (there is a limit to what we can afford). Since a continuous function always has a maximum value on a compact set (Weistrass), the consumer maximization problem is guaranteed to have a solution. Point 1) Observe that:
$$\{x \in \Re^L_+ : \alpha px \leq \alpha w\} = \{x \in \Re^L_+ : px \leq w\}.$$
For point 2) consider a point that is optimal which is below the budget constraint. Then there exsits some epsilon greater than zero such that a ball around the optimal point with radius epsilon will contain another point which the consumer will strictly prefer (and can afford). This is by local non-satiation. Thus we have a contradiction. Point 3) comes directly from the graph with a flat section of the indifference curve. If two points on are this flat section, then the convex combination is both in the set and affordable. Point 4) If the optimal point was not a singleton, then since the preferences are strictly convex, two points which are optimal will have a convex combination which is strictly prefered to either of the points. This is a contradiction to the original point being optimal. This is also clear from the graph.

3 Lecture 3: September 7, 2004

3.1 Utility Maximization Problem

- Consider a utility function: $u(\cdot)$, wealth: $w > 0$, a commodity vector: $x \in X$, and a price vector, $p >> 0$. The consumer's maximization problem is:

$$v(p, w) = max_{x \in X} u(x),$$

$$s.t. \;\; px \leq w.$$

The function $v(p, w)$ is called the value function or the indirect utility function. If we formulate the problem in this way, $v(p, w)$ is the value of the utility function (the range) at the optimum. We could also write:

$$x(p, w) = arg\ max_{x \in X} u(x),$$

$$s.t. \;\; px \leq w.$$

And $x(p, w)$ is now the Walrasian Demand Corrspondence. $x(p, w)$ is the value of the maximizer at the optimum (the domain of the objective function).

- Set up the lagrangian:

$$L = u(x) + \lambda(w - px).$$

- Kuhn Tucker FONCs:

$$\nabla u(x^*) \leq \lambda p.$$

$$x^*[\nabla u(x^*) - \lambda p] = 0.$$

Where this second condition is complementary slackness. Complementary slackness says that we have 2 inequalities and at least one of them must bind.

- The FOCs are equivalent to:

$$\frac{\partial u(x^*)}{\partial x_l} \leq \lambda p_l \text{ and with equality if } x_l^* > 0.$$

- SOC: The hessian matrix, $D^2 u(x^*)$ must be negative semidefinite.

- See graph G-3.1 which shows why we need the complementary slackness condition. If the optimal value is actually on the boarder then the derivative may not be zero. Hence either the tangent is horizontal or the value of the optimizer must be 0.

- See graphs G-3.2 and G-3.3 which shows and interior solution and a corner solution. At an interior solution:

$$\frac{\partial u(x^*)}{\partial x_1} * \frac{1}{p_1} = \frac{\partial u(x^*)}{\partial x_2} * \frac{1}{p_2} = \lambda.$$

This says that the marginal utility of consuming an additional unit of each good must be equal and this is also equal to λ, the shadow price of the constraint, or in this problem, the Marginal Utility of Wealth. At the corner solution in G-3.3,

$$\frac{\partial u(x^*)}{\partial x_1} * \frac{1}{p_1} > \frac{\partial u(x^*)}{\partial x_2} * \frac{1}{p_2}.$$

But the consumer cannot increase his consumption of x_1 anymore. He would actually prefer to consumer negative amounts of x_2.

- **Theorem of the Maximum**. (MWG p. 963) Consider the following maximization problem:

$$max_{x \in \Re^N} f(x, \alpha),$$
$$s.t. \;\; x \in C(\alpha).$$

Where α is just a parameter. What happens to the optimal solution x^* when we change α? The theorem (Berge) states: Suppose $f(\cdot, \alpha)$ is continuous and $C(\alpha)$ is non-empty and compact $\forall\ \alpha$. Further suppose $f(\cdot, \alpha)$ and $C(\alpha)$ vary continuously with α. Then the arg max correspondence is upper hemicontinuous in α and the value function is continuous in α.
A correspondence $H(\cdot)$ is upper hemi-continuous if $x_k \to x$, $\alpha_k \to \alpha$, and $x_k \in H(\alpha_k)$ $\forall\ k$ IMPLIES:

$$x \in H(\alpha).$$

We can also say that H has a closed graph and images of compact sets are bounded. So see graph G-3.4 for a upper hemicontinuous correspondence which is NOT continuous. x_1^* is a solution at each stage (each change in α), but in the limit, there is an additional maximizer, x_2^*. This means that H is upper hemi-continuous (additional maximizers are ok as long as you don't lose any) but not continuous (the set of maximizers has changed). Note that $v(x^*)$ is continuous at all stages since $v(x_1^*) = v(x_2^*)$. So the arg max is upper hemi-continuous an the value function is continuous.

- In the consumer's maximization problem, if we consider the price vector, p, as our parameter, the theorem of the maximum says that the indirect utility function is continuous in p and the walrasian demand correspondence is upper hemi-continuous in p (and it will be continuous as long as preferences are strictly convex).

4 Lecture 4: September 9, 2004

4.1 Homogeneity and Eulers

- **Definition:** A function $f(x_1, x_2, \ldots, x_N)$ is homogeneous of degree r (for any integer r) if for every $t > 0$,

$$f(tx_1, tx_2, \ldots, tx_N) = t^r f(x_1, x_2, \ldots, x_N).$$

- **Definition:** Suppose $f(x)$ is homogeneous of degree r and differentiable. Then evaluated at a point $(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N)$ we have:

$$\sum_{n=1}^{N} \frac{\partial f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N)}{\partial x_n} \overline{x}_n = r f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N).$$

Or in matrix notation:

$$\nabla f(\overline{x}) \cdot \overline{x} = r f(\overline{x}).$$

- **Proof:** Since f is homogeneous of degree r:

$$f(t\overline{x}_1, t\overline{x}_2, \ldots, t\overline{x}_N) = t^r f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N).$$

$$f(t\overline{x}_1, t\overline{x}_2, \ldots, t\overline{x}_N) - t^r f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N) = 0.$$

Differentiate with respect to t:

$$\frac{\partial f(t\overline{x}_1, t\overline{x}_2, \ldots, t\overline{x}_N)}{\partial x_1} \overline{x}_1 + \cdots + \frac{\partial f(t\overline{x}_1, t\overline{x}_2, \ldots, t\overline{x}_n)}{\partial x_N} \overline{x}_N - rt^{r-1} f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N) = 0.$$

Evalute at $t = 1$,

$$\sum_{n=1}^{N} \frac{f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)}{\partial x_n} \overline{x}_n - r f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N) = 0.$$

$$\sum_{n=1}^{N} \frac{f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n)}{\partial x_n} \overline{x}_n = r f(\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_N).$$

QED.

- Corollary 1: If f is homogeneous of degree 0, (h.o.d. 0),

$$\nabla f(\overline{x}) \cdot \overline{x} = 0.$$

- Corollary 2: If f is h.o.d. 1,

$$\nabla f(\overline{x}) \cdot \overline{x} = f(\overline{x}).$$

4.2 Matrix Notation

- Assuming the Walrasian Demand is a function (ie, if the utility function is strictly quasi-concave), we have the demand vector:

$$x(p,w) = \begin{bmatrix} x_1(p,w) \\ x_2(p,w) \\ \vdots \\ x_L(p,w) \end{bmatrix}.$$

- Wealth Effects:

$$D_w x(p,w) = \begin{bmatrix} \dfrac{\partial x_1(p,w)}{\partial w} \\ \dfrac{\partial x_2(p,w)}{\partial w} \\ \vdots \\ \dfrac{\partial x_L(p,w)}{\partial w} \end{bmatrix}.$$

- Price Effects:

$$D_p x(p,w) = \begin{bmatrix} \dfrac{\partial x_1(p,w)}{\partial p_1} & \cdots & \dfrac{\partial x_1(p,w)}{\partial p_L} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial x_L(p,w)}{\partial p_1} & \cdots & \dfrac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}.$$

- **Proposition 2.E.1** says that the walrasian demand satisfies:

$$\sum_{k=1}^{L} p_k \frac{\partial x_l(p,w)}{\partial p_k} + w \frac{\partial x_l(p,w)}{\partial w} = 0 \;\; for \; l = 1...L.$$

  Or in matrix notation:

$$(D_p x(p,w))p + (D_w x(p,w))w = 0.$$

  This is showing how the demand for one good changes as all the other prices change. Proof follows directly from eulers equation above noting that $x(p,w)$ is h.o.d. 0 in (p,w).

- Note that the price elasticity of demand for good l with respect to price k is defined as :

$$\varepsilon_{lk} = \frac{\partial(log\ x_l)}{\partial(log\ p_k)} = \frac{\partial x_l}{\partial p_k}\frac{p_k}{x_l}.$$

  And the elasticity of demand for good l with respect to wealth w is:

$$\varepsilon_{lw} = \frac{\partial(log\ x_l)}{\partial(log\ w)} = \frac{\partial x_l}{\partial w}\frac{w}{x_l}.$$

So we can rewrite proposition 2.E.1 as :

$$\sum_{k=1}^{L} \varepsilon_{lk} + \varepsilon_{lw} = 0 \;\; for \; l = 1...L.$$

- **Proposition 2.E.2** (Cournot Aggregation). $x(p, w)$ has the property for all (p, w):

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p, w)}{\partial p_k} + x_k(p, w) = 0 \;\; for \; k = 1...L.$$

  Or in matrix notation:

$$p \cdot D_p x(p, w) + x(p, w)^T = 0^T.$$

  The proof follows by differentiating Walras law wrt prices. This is showing how the demand for all goods changes as the price of one good changes. If you increase the price of one good by one percent, what happens to the demand for all other goods? Since Walras law still holds, aggregate demand must fall.

- Define the budget share of consumer's expenditure on good l as :

$$b_l(p, w) = \frac{p_l x_l(p, w)}{w}.$$

  Thus rewrite the proposition as:

$$\sum_{l=1}^{L} b_l \varepsilon_{lk} + b_k = 0.$$

- **Proposition 2.E.3** (Engle Aggregation). $x(p, w)$ has the property for all (p, w):

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p, w)}{\partial w} = 1.$$

  Or in matrix notation:

$$p \cdot D_w x(p, w) = 1.$$

  The proof follows from differentiating Walras law wrt w. In elasticities:

$$\sum_{l=1}^{L} b_l \varepsilon_{lw} = 1.$$

  Which says :"The weighted sum of the wealth elasticities is equal to one." If you know the wealth effects on $L-1$ of the goods, the L^{th} wealth effect is automatically implied.

4.3 Properties of the Indirect Utility Function

- **Proposition 3.D.3** (MWG 56-57) If $u(\cdot)$ is a continuous utility function representing any locally non-satiated preference relation, $\succeq$, on the consumption set, $X \in \Re^L_+$, then the indirect utility function, $v(p, w)$, or the value function of the consumer's utility maximization problem, has the following properties:
  - Homogeneous of degree 0 in (p, w).
  - Strictly increasing in w and weakly decreasing in p_l. This is just because you might not consumer good x_l to begin with so its price does not effect your utility.
  - Quasi-Convex in (p, w).
  - Continuous in (p, w).
- **Definition:** Quasi-Convex.
  Definition a:
  $$\text{The set } \{(p, w) : v(p, w) \leq \overline{v}\} \text{ if convex for any } \overline{v}.$$
  Definition b:
  $$\text{If } (p', w') = \alpha(p_1, w_1) + (1 - \alpha)(p_2, w_2) \text{ for } \alpha \in [0, 1],$$
  $$\text{then } v(p', w') \leq max\{v(p_1, w_1), v(p_2, w_2)\}.$$
  See G-4.1 in notes.
- Proof that the indirect utility function is quasi-convex. Consider 2 price/wealth combinations (p_1, w_1) and (p_2, w_2) and the convex combination:
  $$(p', w') = \alpha(p_1, w_1) + (1 - \alpha)(p_2, w_2) \ \ for \ \alpha \in [0, 1].$$
  Consider any x that is affordable at (p', w'):
  $$p'x \leq w'.$$
  Substitute the definition of p' and w':
  $$\underbrace{[\alpha p_1 + (1 - \alpha)p_2]}_{p'} x \leq \underbrace{\alpha w_1 + (1 - \alpha)w_2}_{w'}.$$
  This step is a bit tricky because we initially took the convex combination of a pair of variables (p and w) but we can separate them like we do here because the indirect utility function is homogeneous of degree 0 in (p, w). Thus
  $$\alpha p_1 x + (1 - \alpha)p_2 x \leq \alpha w_1 + (1 - \alpha)w_2.$$
  So it is easy to see that:
  $$\alpha p_1 x \leq \alpha w_1 \quad AND/OR \quad (1 - \alpha)p_2 x \leq (1 - \alpha)w_2.$$

$$p_1 x \leq w_1 \quad AND/OR \quad p_2 x \leq w_2.$$

At least one of these must hold, if they were both strictly greater, we would violate the original inequality. If $p_1 x \leq w_1$, x is in the budget set for (p_1, w_1). If $p_2 x \leq w_2$ then x is in the budget set for (p_2, w_2). Note that $v(p, w)$ is the value function of the consumer's maximization problem which defines the maximum attainable utility. Thus if x is affordable at either (p_1, w_1) or (p_2, w_2), it must be that:

$$u(x) \leq v(p_1, w_1) \quad AND/OR \quad u(x) \leq v(p_2, w_2).$$

Which implies:

$$u(x) \leq max\{v(p_1, w_1), v(p_2, w_2)\}.$$

But x was just some consumption bundle affordable at (p', w'). If we were to choose the optimal bundle at (p', w'), then we have:

$$v(p', w') \leq max\{v(p_1, w_1), v(p_2, w_2)\}.$$

Which is precisely the definition of quasi-convexity.

4.4 Examples of Preferences / Utility Functions

- Leontief Preferences: Fixed coefficients of consumption:

$$u(x_1, x_2) = min\{x_1, \beta x_2\}, \quad \beta > 0.$$

  See Graph G-4.2. The left shoe/right shoe is the obvious example. Indifference curves are corners.

- Homothetic Preferences: all indifference curves are related by proportionate expansion. Thus:

$$x \sim y \iff \beta x \sim \beta y \text{ for every } \beta > 0.$$

  G-4.2 is also homothetic. See G-4.3 for another example. If you can take an indifference curve and multiply it by a constant and end up on another indiffererence curve (for all bundles), then preferences are homothetic. A continuous preference relations, $\succeq$, on $X \in \Re^L_+$ is homothetic implies that $\succeq$ admits a utility function that is homogeneous of degree 1. Consider Cobb-Douglas:

$$u(x_1, x_2) = x_1^{\gamma} x_2^{1-\gamma}.$$

  This utility function is homogeneous of degree 1 (and also homothetic). Note we can take a monotonic transformation of this utility function and lose homotheticity. (Say add 1).

- Quasi-Linear Preferences. We say that preferences are Q-linear with respect to a good and WLOG, let this be good 1. Denote the consumption set as $X = \Re \; x \; \Re^{L-1}_+$, so,

$$x_1 \in \Re,$$

$$x_l \in \Re_+ \; for \; l = 2, \ldots, L.$$

Preferences are Q-linear if:

- 1) All indifferences curves are parallel displacements of each other. So,

$$x \sim y \Longrightarrow x + (\beta, 0, 0, \ldots, 0) \sim y + (\beta, 0, 0, \ldots, 0) \;\; \forall \, \beta.$$

- 2) Good 1 is desirable:

$$x + (\beta, 0, 0, \ldots, 0) \succ x, \; \forall \, \beta > 0.$$

So the indifference curves must be parallel shifts of each other (see G-6.1). Characterisations of Q-linear preferences: a rational and continuous preference relation, $\succeq$, on $X = (-\infty, \infty) \; x \; \Re_+^{L-1}$ is Q-linear if $\succeq$ admits a utility function of the form:

$$u(x_1, x_2, \ldots, x_L) = x_1 + \phi(x_2, x_3, \ldots, x_L).$$

So $u(\cdot)$ is linear in x_1.

5 Lecture 5: September 14, 2004

- Expenditure Minimization Problem:

$$e(p,u) = min_{x \in X}\{p \cdot x\},$$

subject to:

$$u(x) \geq u.$$

Here $e(p,u)$ is the expenditure function which is the minimum amount of money required to attain a given level of utility. We can also rephrase the problem as:

$$h(p,u) = arg\ min_{x \in X}\{p \cdot x\},$$

subject to:

$$u(x) \geq u.$$

Here $h(p,u)$ is the hicksian demand correspondence which says how much of each good do you purchase. Like the walrasian demand function, $x(p,w)$, the only difference is x depends on wealth, and h depends on utility. See graph G-5.1 for a picture of these two problems: Utility Maximization Problem (UMP) and the Expenditure Minimization Problem (EMP).

- Proposition 3.E.3 (MWG pg 61). Suppose $u(\cdot)$ is a continuous utility function representing any locally non-satiated preference relation, $\succeq$ on the consumption set $X \in \Re^L_+$ and u is any attainable utility level. The the hicksian demand correspondence, $h(p,u)$, exhibits the following properties:

  - 1) Homogeneous of Degree 0 in prices:

  $$h(\alpha p, u) = h(p,u)\ \forall\ p \in \Re^L_+,\ \text{attainable } u,\ \alpha > 0.$$

  - 2) No Excess Utility:

  $$u(x) = x\ \forall\ x \in h(p,u).$$

  So this just says the contraint binds (from local non-satiation).
  - 3) If $\succeq$ is convex, then $h(p,u)$ is a convex set.
  - 4) If $\succeq$ is strictly convex, then $h(p,u)$ is a single-valued function.

- Proofs:
Property 1: Minimizing αpx over $\{x \in X : u(x) \geq u\}$ is equivalent to minimizing px over $\{x \in X : u(x) \geq u\}$.
Property 2: Suppose not. Then there exists $x \in h(p,u)$ s.t. $u(x) > u$. But then by continuity, there exists $\alpha < 1$ s.t. $u(\alpha x) > u$. Observe $p\alpha x < px$, which contradicts that x is a solution to the EMP.
Property 3: If $\succeq$ is convex, let $x, y \in h(p,u)$. x and y must both be solutions to the EMP and thus they cost the same amount. Since X is convex, $\alpha x + (1-\alpha)y$ costs the same as x or y so $\alpha x + (1-\alpha)y \in h(p,w)$ for all $\alpha \in [0,1]$ as desired.

Property 4: If $\succeq$ is strictly convex, suppose $x, y \in h(p, u)$ where $x \neq y$. By (3), $\alpha x + (1-\alpha)y \in h(p, u)$, for any $\alpha \in (0, 1)$ but then $\alpha x + (1-\alpha)y \succ x$, contradicting (2).

- Proposition 3.E.2 (MWG pg 59-60). Suppose $u(\cdot)$ is a continuous utility function representing any locally non-satiated preference relation, $\succeq$ on the consumption set $X \in \Re^L_+$ and u is any attainable utility level. Then the expenditure function, $e(p, u)$, exhibits the following properties:
  - 1) Homogeneous of Degree 1 in prices.
  - 2) Strictly increasing in u and weakly increasing in p_l (may have no effect if you don't consume good l).
  - 3) Concave in p.
  - 4) Continuous in p and u.
- Proof of property 3: Fix a level of utility, $\overline{u}$. Consider prices p_1, p_2, and the convex combination:

$$p' = \alpha p_1 + (1-\alpha)p_2, \quad \alpha \in [0, 1].$$

Suppose x' is in the arg min of the EMP where prices are p'. So x' solves the expenditure minimization problem at prices p'. So by definition:

$$u(x') \geq \overline{u}.$$

Or:

$$\begin{aligned} e(p', \overline{u}) &= p'x'. \\ &= \alpha p_1 x' + (1-\alpha)p_2 x'. \\ &\geq \alpha e(p_1, \overline{u}) + (1-\alpha)e(p_2, \overline{u}). \end{aligned}$$

Which is the definition of concavity. Note the last line follows from the fact that if x' is only optimal at p', then any other set of prices (p_1 or p_2) along with x' should lead to at least as much expenditure as finding the optimal bundles say, x_1 and x_2, at prices, p_1 and p_2.

5.1 Duality Relationships between UMP and EMP

- Identities:
  - 1) $e(p, v(p, w)) = w$.
  - 2) $v(p, e(p, u)) = u$.
  - 3) $h(p, u) = x(p, e(p, u))$.
  - 4) $x(p, w) = h(p, v(p, w))$.

- So identity 1 follows from the idea that if you start with wealth, w, and prices, p, fixed, you can find the value function that maximizes your utility, $v(p, w)$, or the indirect utility function. Plugging this into your expenditure function at prices p, should get you right back where you started with your initial wealth since maximizing utility and minimizing expenditure yield the same solution.

- Identity 2 says that for a given utility level u, we find the minimum expenditure required to reach that level. Then the value of that expenditure $v(p, e(p, u))$ gives us back our original level of utility, u.

- Note that the hicksian demand functions are often refered to as compensated demands. If we are holding wealth fixed and changing prices, consider the partial of the walrasian demand. If we are holding utility fixed and changing prices, consider the partial of the hicksian demands (note that wealth would have to implicitly change for utility to remain the same as prices change - hence the "compensated demand."

5.2 Envelope Theorem

- Consider the maximization problem:

$$V(a) = max_{x \in \Re^n} f(x, a),$$

such that:

$$g_1(x, a) = 0,$$
$$g_2(x, a) = 0,$$
$$\dots$$
$$g_m(x, a) = 0.$$

Where a is a parameter of the function. Assume $V(a)$ is differentiable and write the lagrangian as:

$$\mathcal{L}(x, \lambda, a) = f(x, a) + \sum_{m=1}^{M} \lambda_m g_m(x, a).$$

Then, the envelope theorem says:

$$\frac{\partial V(a)}{\partial a_j} = \left.\frac{\partial \mathcal{L}(x(a), \lambda(a), a)}{\partial a_j}\right|_{x(a), \lambda(a)}.$$

In other words, how much does the value function change when you change a parameter? Answer: only look at the direct effect of the parameter through the value function.

- See graphs G-5.2 and G-5.3 in notes which gives some intuition. Because we normally think that around a maximizer, the function is fairly flat, then if you "miss" the solution by a bit, you don't pay too much for your mistake because the range of the "miss" is very small.

- Proposition 3.G.1 (MWG 68-69). Suppose $u(\cdot)$ satisfies the usual properties. Then:

$$h_l(p,u) = \frac{\partial e(p,u)}{\partial p_l} \;\; for\ l = 1 \ldots L.$$

Or,

$$h(p,u) = \nabla_p e(p,u).$$

Proof: In the EMP, the expenditure function is the value function of the minimization:

$$e(p,u) = min_{x \in X}\{p \cdot x\}.$$

The lagrangian of the EMP is:

$$\mathcal{L} = p \cdot x + \lambda(u(x) - u).$$

By the envelope theorem:

$$\frac{\partial e(p,u)}{\partial p_l} = \left.\frac{\partial \mathcal{L}(x,\lambda,p,u)}{\partial p_l}\right|_{x=h(p,u),\lambda=\lambda(p,u)} = \left. x_l \right|_{x=h(p,u)} = h_l(p,u).$$

QED.

6 Lecture 6: September 16, 2004

- Proposition 3.G.4 (Roy's Identity) If $u(\cdot)$ is a utility function with the usual properties, then the walrasian demand is the derivative of the indirect utility function scaled by the marginal utility of wealth:

$$x_l(p,w) = -\frac{\partial v(p,w)/\partial p_l}{\partial v(p,w)/\partial w}, \quad for\ l = 1 \ldots L.$$

  Or in matrix notation:

$$x(p,w) = -\frac{1}{\nabla_w v(p,w)} \nabla_p v(p,w).$$

  Compare this with the previous result regarding the hicksian demand:

$$h(p,u) = \nabla_p e(p,u).$$

  So the difference is that the walrasian demands are a function of wealth so if you change prices, there are WEALTH effects, and hence the scaling. In the hicksian demand, we implicitly allow wealth to vary as we hold utility constant so the result does not need to be scaled.

- Proof. Consider the lagrangian of the utility maximization problem:

$$\mathcal{L} = u(x) + \lambda(w - px).$$

  Where,

$$v(p,w) = max\ \mathcal{L}.$$

  Thus,

$$\frac{\partial v(p,w)}{\partial p_l} = \left.\frac{\partial \mathcal{L}(x,\lambda,p,w)}{\partial p_l}\right|_{x=x(p,w),\lambda=\lambda(p,w)} = -\lambda x_l \Big|_{x=x(p,w),\lambda=\lambda(p,w)}.$$

  But λ is just the marginal utility of wealth:

$$\lambda = \frac{\partial v(p,w)}{\partial w}.$$

  So,

$$\frac{\partial v(p,w)}{\partial p_l} = -\frac{\partial v(p,w)}{\partial w} x_l(p,w).$$

  Or,

$$x_l(p,w) = -\frac{\partial v(p,w)/\partial p_l}{\partial v(p,w)/\partial w}, \quad for\ l = 1 \ldots L.$$

- Recall that if $u(\cdot)$ is quasi-linear, then:

$$u(x_1, x_2, \ldots, x_L) = x_1 + \phi(x_2, x_3, \ldots, x_L).$$

Let $p_1 = 1$ and consider the two good UMP:

$$\mathcal{L} = x_1 + \phi(x_2) + \lambda(w - x_1p_1 - x_2p_2).$$

Then,

$$\frac{\partial \mathcal{L}}{\partial x_1} \Rightarrow 1 - \lambda p_1 = 0.$$

$$\lambda = 1.$$

But since λ is the marginal utility of wealth, we have:

$$\frac{\partial v(p,w)}{\partial w} = 1.$$

So under Q-linear preferences, Roy's Identity reduces to:

$$x(p,w) = -\nabla_p v(p,w).$$

6.1 Price Effects: The Law of Demand

- Prop 3.E.4. Consider a utility function with the usual properties. If $h(p,u)$ is single valued then $h(p,u)$ must satisfy the "compensated law of demand." For all p', p'' pairs,

  $$(p'' - p')[h(p'',u) - h(p',u)] \leq 0.$$

  Proof: For any $p >> 0$, the consumption bundle, $h(p,u)$, is optimal in the EMP such that is achieves a lower expenditure than any other bundle that offers utility u. Thus we have two inequalities:

  $$p''h(p'',u) \leq p''h(p',u),$$

  and,

  $$p'h(p'',u) \geq p'h(p',u).$$

  Subtracting the second inequality from the first:

  $$p''h(p'',u) - p'h(p'',u) \leq p''h(p',u) - p'h(p',u).$$

  Note this makes sense because we are subtracting something large from something small to get something REALLY small and on the right we have something small subtracted from something large so if the first inequality was $\leq$, then after these operations, we must still use the $\leq$. Rearranging:

  $$p''h(p'',u) - p''h(p',u) - p'h(p'',u) + p'h(p',u) \leq 0.$$

  $$(p'' - p')[h(p'',u) - h(p',u)] \leq 0.$$

  As required. QED.

- See graph [G-6.2] for a simple graph of this law of demand. Note when we change one price only, the law of demand says:

$$\frac{\partial h_l(p,u)}{\partial p_l} \leq 0.$$

And this is always true since we have no wealth effects, only substitution effects.

6.2 Price Effects: Substitutes and Complements

- Two goods, l and k, with $l \neq k$ are defined as:

Net Substitutes	if $\frac{\partial h_l}{\partial p_k} \geq 0$	$\forall\, p >> 0,\ \forall\, u$
Net Complements	if $\frac{\partial h_l}{\partial p_k} \leq 0$	$\forall\, p >> 0,\ \forall\, u$
Gross Substitutes	if $\frac{\partial x_l}{\partial p_k} \geq 0$	$\forall\, p >> 0, w > 0$
Gross Complements	if $\frac{\partial x_l}{\partial p_k} \leq 0$	$\forall\, p >> 0, w > 0$

- So if goods are gross substitutes but net complements, it just means that the wealth effect is greater than the substitution effect

6.3 Tâtonnement

- If all goods are gross substitutes for all bidders, then the "Walrasian tâtonnement" yields a competitive equilibrium. Argument is as follows: Consider an initial price vector, $p(0) = (0, 0, \ldots, 0)$. At each time t, assign a Walrasian Auctioneer to ask each consumer $i, (i = 1, \ldots, N)$, to report her demand $x^i(p(t), w)$ and compute aggregate demand as:

$$\overline{x}(p(t)) = \sum_{i=1}^{N} x^i(p(t), w).$$

Then increase the price of each good according to "Walrasian tâtonement":

$$\frac{dp_l(t)}{dt} = \underbrace{\overline{x}_l(p(t)) - S_l}_{Excess\ Demand} \quad for\ l = 1, \dots, L.$$

Where S_l is the supply of good l. Observe that excess demand is always non-negative because:

- 1) At $p(0)$, excess demand must be positive.
- 2) The price of good l stops increasing when excess demand is zero.
- 3) Since all goods are gross substitutes, even if one good "converges" to its optimal price, as the other prices increase, demand (and therefore price) of good l can only rise since the goods are gross substitutes.

• This example is supposed to show a real-world example of what we have been doing and there is an example in the lecture notes about an electricity auction. There is much criticism of the Walrasian Auctioneer because it goes against some of the main foundations of economics. For instance, if agents knew that their consumption decisions affected the prices directly, they could no longer be considered to be operating in a perfectly competitive (price taking) environment. See lecture notes for more or hopefully more will be covered in the next lecture.

7 Lecture 7: September 21, 2004

- Proposition 3.G.2. (MWG 69-70). Consider a utility function $u(\cdot)$ defined as usual. The hicksian demand functions that are derived from this utility function satisfy:

  - (1) $D_p h(p,u) = D_p^2 e(p,u)$.
  - (2) $D_p h(p,u)$ is negative semi-definite.
  - (3) $D_p h(p,u)$ is symmetric.
  - (4) $D_p h(p,u)p = 0$.

  Proof:
  (1) Differentiate the identity: $h(p,u) = \nabla_p e(p,u)$.
  (2) Follows from $e(p,u)$ is concave.
  (3) Since $D_p h(p,u)$ is the second derivative of $e(p,u)$, then (by Young's Theorem):

  $$\frac{\partial^2(\cdot)}{\partial x \partial y} = \frac{\partial^2(\cdot)}{\partial y \partial x}.$$

  (4) Follows directly from the Euler's Equation: $h(p,u) = h(\alpha p, u)$. Differentiating with respect to α:

  $$0 = \frac{\partial h(p,u)}{\partial p} p.$$

- Proposition 3.G.3 (The Slutsky Equation). Define $u(\cdot)$ as usual. Then:

  $$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} * x_k(p,w), \quad for\ l,k = 1,\ldots,L.$$

  Or in matrix notation:

  $$D_p h(p,u) = D_p x(p,w) + D_w x(p,w) x(p,w)^T.$$

  Proof:
  Start with the identity:

  $$h_l(p,u) = x_l(p, e(p,u)).$$

  Differentiate with respect to p_k:

  $$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial e(p,u)} * \frac{\partial e(p,u)}{\partial p_k}.$$

  Note that $w = e(p,u)$ and Prop 3.G.1 says $\frac{\partial e(p,u)}{\partial p_k} = h_k(p,u) = x_k(p, e(p,u)) = x_k(p,w)$, we have:

  $$\frac{\partial h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} * x_k(p,w).$$

- To interpret Slutsky's equation, it is easier to rearrange the terms as:

$$\frac{\partial x_l(p,w)}{\partial p_k} = \underbrace{\frac{\partial h_l(p,u)}{\partial p_k}}_{Substitution} - \underbrace{\frac{\partial x_l(p,w)}{\partial w} * x_k(p,w)}_{Income}.$$

  So we have the change in the (Walrasian) demand for good l from a change in the price of good k decomposed into a substitution effect and an income effect. Note the income effect is "weighted" by the quantity of good k that the agent actually consumes. So if the agent consumes more, the price change effects him more.

- Corollary: If $x(p,w)$ is a Walrasian Demand derived from the usual $u(\cdot)$, then the Slutsky Substitution Matrix defined below must be negative semi-definite and symmetric.

$$D_p h(p,u) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} + \frac{\partial x_1(p,w)}{\partial w} * x_1(p,w) & \dots & \frac{\partial x_1(p,w)}{\partial p_L} + \frac{\partial x_1(p,w)}{\partial w} * x_L(p,w) \\ \vdots & \ddots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} + \frac{\partial x_L(p,w)}{\partial w} * x_1(p,w) & \dots & \frac{\partial x_L(p,w)}{\partial p_L} + \frac{\partial x_L(p,w)}{\partial w} * x_L(p,w) \end{bmatrix}.$$

- Proposition. For any good l, there exists a good k such that good l and good k are net substitutes.
  Proof: This follows from 3.G.2 which says $D_p h(p,u)p = 0$, or the product of the Slutsky Substitution Matrix and the price vector is 0, and proposition 3.E.4 which says $\frac{\partial h_l(p,u)}{\partial p_l} \leq 0$. So think of one row of the Slutsky Substitution matrix. If the element corresponding to the partial of the hicksian with respect to its own price is non-positive, and the whole thing, when multiplied by p, is equal to zero, then there must exist some good k such that $\frac{\partial h_l(p,u)}{\partial p_k} \geq 0$, or l and k are NET substitutes.

7.1 Integrability

- When can a demand function be rationalized? That is, under what conditions must $x(p,w)$ have in order to guarantee that $x(p,w)$ is derived from utility maximizing behavior? Or more precisely, what conditions must $x(p,w)$ satisfy in order to guarantee that there exists an increasing quasiconcave utility function $u(\cdot)$ such that $x(p,w)$ is the Walrasian demand function obtained from $u(\cdot)$?

- To answer this question, we first need to define the concept of "Path Independence." This is a condition that the line integral along any path from a point A to a point B gives the same value. Think of climbing a mountain from a point at the bottom to a point at the top B and noting that the path you take will not effect you reaching the summit. Let $f(x) : \Re^n \mapsto \Re^n$ and $t(z) : [0,1] \mapsto \Re^n$. Then,

$$\int_c f(x)dx = \underbrace{\int_0^1 f(t(z)) \cdot t'(z)dz}_{Line\ Integral} = f(B) - f(A).$$

- Another way of saying this is that the line integral along any closed path ($A \to B \to A$) equals zero. See G-7.1.

- From this, we have the following result: **Theorem** (Fundamental Theorem of Calculus of Line Integrals). Let C be any piecewise smooth curve from a point A to a point B. Then the line integral of $\nabla\phi$ is path independent and:

$$\int_c \nabla\phi dp = \phi(p_B) - \phi(p_A).$$

- Also, we get another **Theorem**: Given any real-valued function $\phi(\cdot)$, define $f = \nabla\phi$. Then:

$$\frac{\partial f_j}{\partial p_i}(\overline{p}) = \frac{\partial f_i}{\partial p_j}(\overline{p}) \ \ \forall\, \overline{p}.$$

  Proof: The second derivative is independent of the order of differentiation.

$$\frac{\partial^2\phi}{\partial p_i p_j}(\overline{p}) = \frac{\partial^2\phi}{\partial p_j p_i}(\overline{p}) \ \ \forall\, \overline{p}.$$

- Conversely, we have another result. **Theorem**: Given any function $f(\cdot) = (f_1, \ldots, f_L)$ such that

$$\frac{\partial f_j}{\partial p_i}(\overline{p}) = \frac{\partial f_i}{\partial p_j}(\overline{p}) \ \ \forall\, \overline{p},$$

  then there exists a real-valued function $\phi(\cdot)$ such that $f = \nabla\phi$.

- So if $h(p,u)$ has a symmetric substitution matrix, then there exists a function $e(p,w)$ such that $h(p,u) = \nabla_p e(p,w)$. This is a sufficient condition.

- Now we come to the Main Proposition on Integrability. **Proposition:** A continuously-differentiable function $x : \Re^{L+1}_{++} \mapsto \Re^L_+$, which satisfies Walras Law and such that $D_p h(p,u)$ is symmetric and negative semi-definite, IS the demand function generated by *some* increasing, quasi-concave, utility function.

- So the conditions on $x(p,w)$ such that there exists a utility function with $x(p,w) = arg\ max\ u(x)$ s.t. $px \leq w$ are:
  - (1) Continuously Differentiable.
  - (2) Satisfies Walras' Law.
  - (3) $D_p h(p,u)$ is Symmetric and Negative Semi-Definite.

- The symmetry of the slutsky substitution matrix gives us the existence of the expenditure function and the negative semi-definitness gives us the concavity of the expenditure function.

- Sketch Proof of this main proposition: First recover $e(p, u)$ from $x(p, w)$ (observed). For $L = 2$ and $p_2 = 1$,
$$\frac{de(p_1)}{dp_1} = x_1(p_1, e(p_1)).$$
Initial condition $w_0 = e(p_1^0)$. So for more than two goods, we just have the L partials to solve. Again, the solution will exist (an expenditure function can be found) so long as the slutsky substitution matrix is symmetric. The second step is to recover the preference relation, $\succeq$, from $e(p, u)$. Given an expenditure function $e(p, u)$, define a set:
$$V_u = \{x : p \cdot x \geq e(p, u) \; \forall \; p >> 0\}.$$
Note that elements of V_u are "at least as good as x" so if we do this for all the possible consumption bundles, we have defined our preference relation, $\succeq$. And we're golden.

8 Lecture 8: September 23, 2004

8.1 Welfare Evaluation

- What is the effect on the consumer of a change in prices (say from p^0 to p^1). The idea of the answer involves the distance from $u^0 = v(p^0, w)$ to $u^1 = v(p^1, w)$ as measured in monetary units.

- Consider the following money metric indirect utility function:

$$e(\overline{p}, v(p, w)), \quad \overline{p} >> 0.$$

  This gives us the wealth required to reach the utility level $v(p, w)$ at prices $\overline{p}$. So all we have done is apply a monotonic tranformation to the indirect utility function which should maintain the same preferences (note that $e(p, u)$ is strictly increasing in u), and we have something that is denominated in say, dollars, which represents utility. See G-8.1. To measure changes in wealth, we need to evaluate at either the old or new prices. This involves shifting back the new budget constraint to the old indifference curve (or vice versa) and looking at the vertical distance between the two parallel lines (assuming $p_2 = 1$).

- Consider the two interesting cases of welfare changes:
  - Equivalent Variation (EV) - Old Prices:

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w.$$

  - Compensating Variation (CV) - New Prices:

$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0).$$

- See G-8.2 for the EV and G-8.3 for the CV. Note we always assume the price of good 1 changes (falls in the graphs), and that the price of good 2 is constant and equal to 1.

- So the EV can be thought of as the dollar amount that the consumer would be indifferent about accepting in lieu of the price change. Hence:

$$EV = e(p^0, u^1) - e(p^0, u^0).$$

$$EV = e(p^0, u^1) - w.$$

$$w + EV = e(p^0, u^1).$$

  Apply $v(p^0, \cdot)$,

$$v(p^0, w + EV) = v(p^0, e(p^0, u^1)).$$

$$v(p^0, w + EV) = u^1.$$

- Similarly, the CV can be thought of as the net revenue of a planner who must compensate the consumer for a price change after it occurs, bringing the consumer back to her original utility level. Hence:

$$CV = e(p^1, u^1) - e(p^1, u^0).$$

$$CV = w - e(p^1, u^0).$$
$$w - CV = e(p^1, u^0).$$

Apply $v(p^1, \cdot)$,

$$v(p^1, w - CV) = v(p^1, e(p^1, u^0)).$$
$$v(p^1, w - CV) = u^0.$$

- So which measure should we use? If we're interested in some compensation scheme at a new set of prices, use the CV. However, if we are interested in consumer's willingness to pay for something, use EV because 1) EV is measured at old (current) prices so it is easier to evaluate the effects of a policy or project and 2) comparing the outcomes from several projects is easier because all the prices will be in today's dollars while each project would generate a new CV at new prices so they would not be comparable.

- Carrying the analysis further, recall shepards lemma:

$$\frac{\partial e(p, u)}{\partial p_1} = h_1(p, u).$$

The hicksian demand for good 1 is equal to the partial of the expenditure function with repect to the price of good 1. Now consider that the price of good 1 changes while all the other goods remain at the same price. So,

$$p^0 = (p_1^0, p_{-1}), \quad and \quad p^1 = (p_1^1, p_{-1}).$$

And keep in mind that $w = e(p^0, u^0) = e(p^1, u^1)$. And now rewrite EV as:

$$\begin{aligned} EV(p^0, p^1, w) &= e(p^0, u^1) - e(p^0, u^0) \\ &= e(p^0, u^1) - e(p^1, u^1) \\ &= e(p, u^1)\Bigg|_{p=p^1}^{p=p^0} \\ &= \int_{p_1^1}^{p_1^0} \frac{\partial e(p, u^1)}{\partial p_1} dp_1 \\ &= \int_{p_1^1}^{p_1^0} h_1(p_1, p_{-1}, u^1) dp_1 \end{aligned}$$

- Equivalently,

$$\begin{aligned} CV(p^0,p^1,w) &= e(p^1,u^1)-e(p^1,u^0) \\ &= e(p^0,u^0)-e(p^1,u^0) \\ &= e(p,u^0)\Big|_{p=p^1}^{p=p^0} \\ &= \int_{p_1^1}^{p_1^0} \frac{\partial e(p,u^0)}{\partial p_1} dp_1 \\ &= \int_{p_1^1}^{p_1^0} h_1(p_1,p_{-1},u^0)dp_1 \end{aligned}$$

- See graphs G-8.4 for the picture of the EV as the area under the hicksian demand for good 1 (at utility level 1) between the two prices. The graph for the CV would be the same but it's the area under the hicksian demand for good 1 at utility level 0.

- In general, by path independence,

$$EV(p^0,p^1,w) = \int_C h(p,u^1)dp,$$

$$CV(p^0,p^1,w) = \int_C h(p,u^0)dp,$$

where C is a curve from p^1 to p^0. Note we used the very specific case of a change in the price of good 1 only above, while here, we generalize for ANY two price vectors.

- Next, define a third measure of consumer welfare, Consumer Surplus (CS):

$$CS = \int_{p_1^1}^{p_1^0} x(p_1,p_{-1},w)dp_1.$$

So CS is the area under the walrasian demand for good 1 at wealth w. See G-8.5 and G-8.6 for a pictures of (X,P) space where the hicksian and walrasian demand curves are plotted together. For normal goods (G-8.5) the hicksian demand curves are steeper than the walrasian demand curve (via Slutsky) while for inferior goods (G-8.6), the hicksian demands are more shallow than the walrasian demand. Hence, as seen in the areas in the graph:

Normal Good: $CV < CS < EV$.

Inferior Good: $EV < CS < CV$.

- Finally, if there are NO INCOME EFFECTS, then CV = CS = EV, as in the case of quasi-linear preferences.

9 Lecture 9: September 28, 2004

9.1 More on Welfare Evaluation

- Example 3.I.1. This example demonstrates the deadweight loss (DWL) associated with a commodity tax versus having a lump sum tax that raised the same amount of revenue.

- Consider a commodity tax on good 1 such that:

$$p^0 = (p_1^0, p_{-1}).$$

$$p^1 = (p_1^0 + t, p_{-1}).$$

Revenues from this tax equal $T = tx_1(p^1)$. The consumer is made worse off provided that the equivalent variation (EV) is less than $-T$, the amount of wealth the consumer loses under the lump sum tax. Recall:

$$EV = e(p^0, u^1) - e(p^0, u^0).$$

Thus,

$$-T - EV = e(p^0, u^0) - e(p^0, u^1) - T.$$

$$-T - EV = e(p^1, u^1) - e(p^0, u^1) - T.$$

$$-T - EV = \int_{p_1^0}^{p_1^0+t} h_1(p_1, p_{-1}, u^1)dp_1 - th_1(p_1^0 + t, p_{-1}, u^1).$$

Bring the second term inside the integral:

$$-T - EV = \int_{p_1^0}^{p_1^0+t} \underbrace{[h_1(p_1, p_{-1}, u^1) - h_1(p_1^0 + t, p_{-1}, u^1)]}_{\geq 0} dp_1 \geq 0.$$

So the difference is non-negative which means:

$$-T - EV \geq 0.$$

$$EV \leq -T.$$

So what the government would have to pay the consumer, the EV, is smaller (more negative) than the lump sum tax so there must be a DWL associated with the commodity tax. This can also be seen using the CV:

$$CV = e(p^1, u^1) - e(p^1, u^0).$$

Thus,

$$-CV - T = e(p^1, u^0) - e(p^1, u^1) - T.$$

$$-CV - T = e(p^1, u^0) - e(p^0, u^0) - T.$$

$$-CV - T = \int_{p_1^0}^{p_1^0+t} h_1(p_1, p_{-1}, u^0)dp_1 - th_1(p_1^0 + t, p_{-1}, u^0).$$

Bring the second term inside the integral:

$$-CV - T = \int_{p_1^0}^{p_1^0+t} [\underbrace{h_1(p_1, p_{-1}, u^0) - h_1(p_1^0 + t, p_{-1}, u^0)}_{\geq 0}]dp_1 \geq 0.$$

So the difference is non-negative which means:

$$-CV - T \geq 0.$$

$$CV \leq -T.$$

Again, a DWL. Thus, see G-9.1 for graphs of the deadweight losses. Note in general, it's the area to the left of the hicksian demands between the two prices less the tax revenue which is the rectangle.

- Another example is a monopoly. See G-9.2. Here, we assume Q-linear preferences so $x(p, w) = h(p, u)$. The DWL of the monopoly price versus the competitive price is shaded in the graph.

- In general, if you are comparing two possible policies which will result in two possible price vectors, p^1 or p^2, then use the EV to compare. Consider:

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0).$$

$$EV(p^0, p^2, w) = e(p^0, u^2) - e(p^0, u^0).$$

So,

$$EV(p^0, p^1, w) - EV(p^0, p^2, w) = e(p^0, u^1) - e(p^0, u^2).$$

So this allows for direct comparison such that "p^1 is better than p^2" $\iff EV(p^0, p^1, w) > EV(p^0, p^2, w)$. With the CV this is impossible because,

$$CV(p^0, p^1, w) - CV(p^0, p^2, w) = e(p^2, u^0) - e(p^1, u^0),$$

and with two different price vectors, we can't say anything more about which is preferable.

- Thus, EV is a transitive measure of welfare while CV may in intransitive. This also means that EV is a valid money-metric indirect utility function while CV is not.

9.2 Revealed Preference

- So far we have used a preference based approach to demand instead of a choice based one. To use actual choices, we develope the Weak Axiom of Revealed Preferences (WARP).

- **Definition:** (2F1) $x(p,w)$ satisfies the WARP if, for any two price wealth pairs (p^1, w^1) and (p^2, w^2):

$$p^1 \cdot x(p^2, w^2) \leq w^1, x(p^1, w^1) \neq x(p^2, w^2) \Rightarrow p^2 \cdot x(p^1, w^1) > w^2.$$

  See graphs G-9.3 thru G-9.5 for a graphical interpretation. Basically, we have to have choice consistency. If $x(p^2, w^2)$ is affordable at prices p^1 and wealth w^1, but we still choose $x(p^1, w^1)$, then it must be the case that at prices p^2, and wealth w^2, the bundle $x(p^1, w^1)$ is not affordable since we would have chose it over $x(p^2, w^2)$ because it was prefered at prices p^1 and wealth w^1.

- The WARP is equivalent to the compensated law of demand. Proposition 2.F.1 says that if $x(p,w)$ is h.o.d. 0 in (p,w) and satisfies Walras law,

$$(p^2 - p^1) \cdot [x(p^2, w^2) - x(p^1, w^1)] \leq 0.$$

  This inequality is strict if the bundles are different. Note when we say compensated, we mean that wealth is not held constant here – it does not have something to do with hicksian demands.

- Moreover, the compensated law of demand implies that the substition matrix is negative semi-definite. Thus, if $x(p,w)$ satisfies h.o.d. 0, walras law, and WARP, then the Slutsky matrix is negative semi-definite. So what are we missing? SYMMETRY.

- Definition 3.J.1 introduces the Strong Axiom of Revealed Preferences (SARP) which adds in transitivity of revealed preferences.

- Finally, Proposition 3.J.1 says that if the function $x(p,w)$ satisfies the SARP, then there is a rational preference relation, $\succeq$, that rationalizes $x(p,w)$. In other words, for all (p,w), we have $x(p,w) \succ y$ for every $y \neq x(p,w)$ with $y \in B_{p,w}$. So while WARP lacked symmetry of the slutsky matrix, SARP gives us everything we need including symmetry.

10 Lecture 10: September 30, 2004

10.1 Aggregate Demand

- Denote individual demand:
$$x_i(p, w_i).$$
Where p is a common price vector and wealth is particular to each consumer. Then aggregate demand might be defined as:
$$x(p, w) = \sum_{i=1}^{n} x_i(p, w_i).$$
But generally, not only aggregate wealth, but the distribution of wealth matters in aggregating demand.

- Consider two individuals with wealth w_1 and w_2. They each have a certain income effect for each of the L goods and aggregate wealth, $w = w_1 + w_2$. Now consider two other individuals with wealth:
$$w_1' = w_1 + \Delta.$$
$$w_2' = w_2 - \Delta.$$
Note that $w_1' + w_2' = w$, the same aggregate but these two individuals might have different wealth effects for the different goods. Thus aggregate demand must take this into account.

- From the above example, it is clear that there is no such thing as an aggregate slutsky equation or an aggregate slutsky substitution matrix. In general, for individuals with strictly convex preferences, the only restrictions on aggregate demand are that it must be hod(0), continuous, and satisfy a version of walras law.

- So what restrictions do we need on aggregate demand such that it is completely characterized by aggregate wealth ? Pretty specific ones:

- Proposition 4B1. Aggregate demand can be expressed as a function of aggregate wealth if and only if all consumers have preferences admitting an indirect utility function of the following form:
$$v_i(p, w_i) = a_i(p) + b(p)w_i.$$
Note that the first term can be different for each individual but the coefficient on the individuals wealth must be the same across individuals. This is called the **Gorman Form Indirect Utility Function.** Examples of preferences which admit a function such as this are:

  - 1) Preferences of all consumers are identical and homothetic.
  - 2) Preferences of all consumers are quasilinear with respect to the same good (NO wealth effects).

- The reasoning behind 4B1 is Roy's identity. Recall that walrasian demand is the partial of the indirect utility with respect to the price divided by the partial with respect to wealth. The denominator must be the same for things to aggregate nicely.

- Finally, if every consumer has homothetic (but different) preferences, then aggregate demand satisfies the WARP. We don't get symmetry and negative semi-definiteness of the substitution matrix though (it doesn't even exist!)

- **Definition** $x_i(p, w_i)$ satisfies the UNcompensated law of demand if:

$$(p_2 - p_1) \cdot [x_i(p^2, w_i) - x_i(p^1, w_i)] \leq 0.$$

  (Strict inequality if $x_i(p^1, w_i) \neq x_i(p^2, w_i)$.)

- Proposition 4C1. If $x_i(p, w_i)$ satisfies the uncompensated law of demand for all consumers i, then so does aggregate demand. Thus aggregate demand satisfies the WARP.

- Proposition 4C2. If $\succeq_i$ is homothetic, then $x_i(p, w_i)$ satisfies the uncompensated law of demand.

10.2 Theory of the Firm

- Production Set Notation. A production set Y is the set of all feasible production plans. This notation allows for multiple outputs and there is no need to have distinct input and output sets.

- Define $F(\cdot)$ as the transformation which satisfies:

$$Y = \{y \in \Re^L : F(y) \leq 0\},$$

  and the boundry of Y is described by $F(y) = 0$. See G-10.1. Note that y_1 and y_2 could be inputs and/or outputs. It just depends on where your point y is. At the point a, y_1 is an output (it's positive) but y_2 is zero. This is not in the production set because you produce something out of nothing (usually not possible!). The point b is in the production set but here, both y_1 and y_2 are inputs and we have NO output ... usually not a very good production plan! In two dimensions, it looks like usually we will be finding production plans (reasonable) in the NW and SE quadrants along the frontier.

- Define the Marginal Rate of Transformation of good l for good k as $y \in Y$ as:

$$MRT_{lk}(y) = \frac{\partial F(y)/\partial y_l}{\partial F(y)/\partial y_k}.$$

  Along the frontier, this is just the slope of $F(y)$. Note depending on which y you evalute this at, you could get either the marginal product of an input or the marginal rate of substitution between two inputs.

- The other notation frequently used in the theory of the firm is Production Function notation. Here, we have a production funciton $f(z)$ defined as the maximum quantity of output, q, that can be produced using a given input vector, z. This restricts the attention to single output and forces us to have distinct input and output sets. See G-10.2.

- Define the Marginal Rate of Technical Substitution (MRTS) as:

$$MRTS_{lk}(z) = \frac{\partial f(z)/\partial z_l}{\partial f(z)/\partial z_k}.$$

Properties Often Assumed of the Production Set, Y

- (1) Non-empty - there exists at least one feasible production plan in Y.
- (2) Closed - the limits of feasible production plans in Y are also in Y.
- (3) No Free Lunch - if $y \in Y$ and $y \geq 0$, then $y = 0$.
- (4) Possibility of Inaction - $0 \in Y$.
- (5) Free Disposal - if $y \in Y$ and $y' \leq y$ then $y' \in Y$.
- (6) Irreversibility - if $y \in Y$ and $y \neq 0$, then $-y \notin Y$ (ie, the production process cannot run in both directions). Note that convex production sets, Y, will satisfy this automatically.
- (7) NonINcreasing Returns to Scale - if $y \in Y$ and $\alpha \in [0, 1]$, then $\alpha y \in Y$. See G-10.3 If y is in Y, then all production plans on the ray connecting y to the origin must be in Y. This is saying that doubling inputs cannot quite double outputs, but the graph is showing that halving inputs will necessarily leave you with at least half the output.

  item (8) NonDEcreasing Returns to Scale - if $y \in Y$ then for all $\alpha \geq 1$, $\alpha y \in Y$. See G-10.4 If y is in Y, then all production plans on the outside part of the ray (B) connecting y to the origin must be in Y. So doubling inputs will leave us with at least as much output.
- (9) Constant Returns to Scale - if $y \in Y$ and $\alpha \geq 0$, $\alpha y \in Y$. We sometimes call Y a cone. See G-10.5.

11 Lecture 11: October 5, 2004

11.1 More Properties of Y

- 10) Additivity (Free entry): if $y, y' \in Y$, then $y + y' \in Y$.
- 11) Convexity: if $y, y' \in Y$ and $\alpha \in [0,1]$, then $\alpha y + (1-\alpha)y' \in Y$.
- 12) Convex Cone: Convexity + CRS.
- See graph G-11.1 for a picture of a production set which exhibits CRS but is NOT convex (it also violates free-disposal).

11.2 Profit Maximization Problem (PMP)

- Define y: feasible production plans (could be inputs or outputs), $F(y)$: transformation function, and a price vector, $p >> 0$.
- The problem of the firm is:
$$\pi(p) = max_y \; p \cdot y,$$
such that,
$$F(y) \leq 0.$$
It can also be written:
$$y(p) = arg \; max_y \; p \cdot y,$$
such that,
$$F(y) \leq 0.$$
Where $\pi(p)$ is the profit function of the firm and $y(p)$ is the supply function (with inputs as negative quantities).
- The one concern with this type of problem is that unlike in the consumer's maximization problem where income was bounded, we have not explicitly assumed that the firm has finite resources. In fact with non-decreasing returns to scale and some y such that $p \cdot y > 0$, this problem is unbounded and the firm should expand output forever.
- Lagrangian:
$$\mathcal{L} = p \cdot y + \lambda(-F(y)).$$
$$\mathcal{L} = p \cdot y - \lambda F(y).$$
- FOC:
$$p_l = \lambda \frac{\partial F(y^*)}{\partial y_l}.$$
Or in matrix notation:
$$p = \lambda \nabla F(y^*).$$

- Production function notation (single output) with a vector of inputs, z, at prices, w, and with one output, p is just a scalar. Problem becomes:

$$max_{z\geq 0}\, pf(z) - w\cdot z.$$

FOC:

$$p\frac{\partial f(z)}{\partial z_l} \leq w_l,$$

with complementary slackness:

$$(p\frac{\partial f(z)}{\partial z_l} - w_l)z_l = 0.$$

Note that the CS condition comes in when the firm would rather consume less than 0 units of the input but cannot.

- Proposition 5C1. Suppose $\pi(\cdot)$ is the profit function and $y(\cdot)$ is the supply correspondence. If Y is closed and satisfies free-disposal, then:
  - (0) $\pi(\cdot)$ is continuous in p (if finite) and $y(\cdot)$ is upper-hemicontinuous in p.
  - (1) $\pi(\cdot)$ is hod 1.
  - (2) $\pi(\cdot)$ is convex.
  - (3) If Y is convex, then

    $$Y = \{y \in \Re^L : p\cdot y \leq \pi(p)\; \forall\, p >> 0\}$$

  - (4) $y(\cdot)$ is hod 0 (if you double both input and output prices, profits remain unchanged).
  - (5) If Y is convex, then $y(p)$ is a convex set for all p. Also if Y is strictly convex, then $y(\cdot)$ is a continuous function of p (single valued).
  - (6) Hotelling's Lemma: If $y(p)$ is single valued then $\pi(\cdot)$ is differentiable at p and:

    $$\nabla\pi(p) = y(p).$$

  - (7) If $y(\cdot)$ is differentiable at p then the substitution matrix:

    $$Dy(p) = \nabla_p^2\pi(p) = \begin{bmatrix} \frac{\partial y_1}{\partial p_1}(p) & \dots & \frac{\partial y_1}{\partial p_L}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial y_L}{\partial p_1}(p) & \dots & \frac{\partial y_L}{\partial p_L}(p) \end{bmatrix},$$

    is symmetric and POSITIVE semi-definite with $Dy(p)\cdot p = 0$.
  - Consequently, we have the Law of Supply:

    $$(y' - y)(p' - p) \geq 0.$$

Or,

$$\frac{\partial y_l(\overline{p})}{\partial p_l} \geq 0.$$

Recall that y is just a feasible bundle in output OR inputs. So for outputs, this has the usual interpretation. But for inputs, it is also valid because inputs are measured in negative numbers so increasing the price of an input still results in the firm reducing their demand for that input.

- Another result is integrability: The functions $y(\cdot)$ are supply functions generated by some convex production set, Y, if they are hod 0 and their substitution matrix is symmetric and postive semi-definite. In other words, y needs to be the gradient of a profit function.

11.3 Cost Minimization Problem (CMP)

- Define z again as an input vector at prices w. $f(z)$ the production function generating output q. The problem is now:

$$C(w, q) = min_{z \geq 0}\; w \cdot z,$$

such that,

$$f(z) \geq q.$$

Or, we also write:

$$z(w, q) = arg\; min_{z \geq 0}\; w \cdot z,$$

such that,

$$f(z) \geq q.$$

Where $C(w, q)$ is the cost function and $z(w, q)$ is the conditional factor demands.

- Proposition 5C2. Suppose that the production function $f(\cdot)$ is continuous and strictly increasing. Then:

$C(w, q)$	$z(w, q)$
(1) hod 1 in w	(1) hod 0 in w
(2) Strictly increasing in q	(2) No excess production
(3) Continuous in (w, q)	(3) Upper hemi-continuous in (w, q)
(4) Concave in w	(4) f: q-concave $\Rightarrow z(w, q)$ is a convex set
	f: strictly q-concave $\Rightarrow z(w, q)$ is single valued.

- Further properties.
  - (1) Shephards Lemma: $z(w, q) = \nabla C(w, q)$.

* (2) The following substitution matrix is symmetric and NEGATIVE semi-definite:

$$\sigma^*(z,q) = \nabla_w z(w,q) = \begin{bmatrix} \frac{\partial z_1}{\partial w_1} & \cdots & \frac{\partial z_1}{\partial w_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial z_L}{\partial w_1} & \cdots & \frac{\partial z_L}{\partial w_L} \end{bmatrix}.$$

* (3) As a consequence of (2), we have:

$$\frac{\partial z_l(w,q)}{\partial w_l} \leq 0 \; \forall \; l.$$

  Or in the less "babyish" form:

$$(z' - z)(w' - w) \leq 0.$$

  Which we might call the "Law of Input Demand."
* (4) If $f(\cdot)$ is hod 1 (CRS), the $C(\cdot)$ and $z(\cdot)$ are hod 1 in q.
* (5) If $f(\cdot)$ is concave, then $C(\cdot)$ is a convex function of q (ie, marginal costs are weakly increasing in q).

12 Lecture 12: October 7, 2004

12.1 Duality

- See G-12.1 Regarding the graph of the isoquant ($f(z) = q$) tangent to the isocost ($c(w', q) = w'z$), we have two regions. It is clear from the diagram that:

$$\{z : f(z) \geq q\} \subseteq \{z : w'z \geq c(w', q)\}.$$

  So the upper contour set of the isoquant is a subset of the upper contour set of the isocost. By definition, any vector of inputs that yields output of at least q will cost at least $c(w', q)$. In other words, anything that costs less than $c(w', q)$ must also produce less than q.

- We can repeat this process by varying the input vector prices (w) and finding the tangential point. (G-12.2) We find that this traces out the isoquant even if we didn't know the isoquant to being with. We fix q and find the minimum cost at prices w of producing that q. Thus, we can derive the isoquant (technology) just from the cost function.

- Formally, the upper contour set of the isoquant is:

$$\bigcap_{w>>0} \{z \in \Re^L_+ : w \cdot z \geq c(w, q)\}.$$

  Or,

$$\{z \in \Re^L_+ : w \cdot z \geq c(w, q) \ \forall \ w >> 0\}.$$

  Equivalently, the production function is defined as:

$$f(z) = max\{q \geq 0 : w \cdot z \geq c(w, q) \ \forall \ w >> 0\}.$$

  Ie, we vary the input price vector and then find the maximium quantity attainable at minimum cost.

- Note that in the previous analysis, we assumed a convex isoquant. See G-12.3 for a picture of a non-convex isoquant. However, even in this case, it is still possible to recreate duality. The envelope formed by the isocost is not the original isoquant, however it is the highest convex and monotonic curve that is weakly below the original isoquant. Also, and most importantly, the areas that we miss (see G-12.4) are not optimal anyway since there are lower cost ways to produce to the same level of output.

- Statement of Duality: Start with a production function, $f(z)$, which is continuous, weakly increasing, and quasi-concave. Let $c(w, q)$ be the cost function implied by $f(z)$. Then:

$$c(w, q) = min_{z \in \Re^L_+}\{w \cdot z : f(z) \geq q\}.$$

Now start with $c(w, q)$ and construct $f(z)$ using duality. Thus,

$$f^*(z) = max\{q \geq 0 : w \cdot z \geq c(w, q) \; \forall \; w >> 0\}.$$

Then $f^*(z) = f(z)$.

- Consider any function, $c(w, q)$, satisfying the usual properties of cost functions. When does a cost function arise from a profit-maximizing firm? When the following are true:
  - (1) $c(w, 0) = 0$.
  - (2) Continuous.
  - (3) Strictly increasing in q.
  - (4) Weakly increasing in w.
  - (5) HOD(1) in w.
  - (6) Concave in w.

  Then $f(z) = max\{q \geq 0 : w \cdot z \geq c(w, q) \; \forall \; w >> 0\}$ is increasing and quasi-concave and the cost function generated by $f(z)$ is $c(w, q)$.

- 2 More Duality Results:
  - (1) Given the expenditure function, $e(p, u)$, we can determine the upper contour set of the indifference curves:

    $$V_{\overline{u}} = \{x \in \Re^L_+ : u(x) \geq \overline{u}\},$$

    by calculating:

    $$\{x \in \Re^L_+ : p \cdot x \geq e(p, \overline{u}) \; \forall \; p >> 0\}.$$

    Or equivalently,

    $$u(x) = max\{\overline{u} > 0 : p \cdot x \geq e(p, \overline{u}) \; \forall \; p >> 0\}.$$

  - (2) Given a profit function, $\pi(p)$, we can determine the production set, Y, by calculating:

    $$Y = \{y \in \Re^L : p \cdot y \leq \pi(p) \; \forall \; p >> 0\}.$$

    We saw this result earlier. Here y is a vector of inputs and outputs (a production plan). So we have the "set of all production plans which provide at most $\pi(p)$ for any given price vector."

- Finally see G-12.5 for a diagram connecting the UMP to the EMP.
  - In the consumer's utility maximization problem, we max $u(x)$ such that $px \leq w$. The maximized function is $v(p, w)$, the indirect utility function, and the argument which maximizes is the walrasian demand, $x(p, w)$ (uncompensated demand).

- In the consumer's expendituve minimization problem, we $min\ px$ such that $u(x) \geq u$. The maximized function is $e(p,u)$, the expenditure function, and the argument which minimizes is the hicksian demand, $h(p,u)$ (compensated demand).
- We can link $v(p,w)$ to $x(p,w)$ using Roy's Identity. Wealth effects come in to play here.
- We can link $e(p,u)$ to $h(p,u)$ just using $h(p,u) = \nabla_p e(p,u)$. No wealth effects to mess things up.
- We can link the derivatives of $x(p,w)$ to $h(p,u)$ using the slutsky equation.
- Finally, we can use duality to go from the expenditure function to the utility function by tracing out the indifference curves at varying price vectors.

13 Lecture 13: October 12, 2004

13.1 More on the Geometry of Cost Curves

- See graphs G-13.1, G-13.2 and G-13.3 for plots of production functions and cost curves for non-sunk and sunk costs. Notice that for non-sunk costs, the supply curve is equal to the MC curve above the AC curve and zero elsewhere. For sunk costs, these should NOT enter the decision process so the supply curve is equal to the marginal cost even below the AC curve.

- See G-13.4 for a graph showing the short run and long run total and average cost curves. Note that in the short run, some inputs may be fixed and hence we have a decision problem in the short run with a constraint (say $z_2 = \overline{z}$), then the short run cost curve must lie above or touching the long run cost curve. The same is true for average costs. Thus the envelope formed by the short run cost curves is the long run cost curve. In the long run, all inputs may vary.

- See G-13.5 for a simple graph of constant returns to scale. Same idea as the other graphs.

Aggregate Supply

- In the theory of the firm, individual supplies depend only on prices (not wealth) so things work much better. In particular, if $y_j(p)$ represents the supply of firm j. The aggregate supply is:

$$y(p) = \sum_{j=1}^{J} y_j(p).$$

The substitution matrix for each individual supply is hod(0), symmetric, and positive semi-definite. These properties also hold for aggregate supply. Consequently, aggregate supply can be rationalized as arising from a single profit maximizing firm whose production set is:

$$Y = Y_1 + Y_2 + \cdots + Y_J.$$

- Proposition 5E1. The aggregate profit attained by maximizing profit separately is the same as that which would be obtained if the production units were to coordinate their actions, when firms are price takers. Thus, we have the law of aggregate supply:

$$(p^1 - p^2) \cdot (y(p^1) - y(p^2)) \geq 0.$$

13.2 Monopoly and Price Discrimination

- There are 4 types of situations that we will examine:
  - (1) Uniform + Linear Pricing (Classic Monopoly).
  - (2) Uniform + Nonlinear Pricing (2nd Degree Price Discrimination).

- (3) Nonuniform + Linear Pricing (3rd Degree Price Discrimination).
- (4) Nonuniofrm + Nonlinear Pricing (1st Degree Price Discrimination).

Classic Monopoly

- Consider a monopolist who faces an inverse demand curve $p(q)$ and has cost function $c(q)$. The problem of the monopolist is:

$$max_{q \geq 0}\ q * p(q) - c(q).$$

- FOC:

$$\underbrace{p(q) + q * p'(q)}_{Marginal\ Revenue} - \underbrace{c'(q)}_{Marginal\ Cost} = 0.$$

- Rearranging:

$$p(q) + q * p'(q) = c'(q).$$
$$p(q)(1 + \frac{q * p'(q)}{p(q)}) = c'(q).$$
$$p(q)(1 + \frac{q * dp}{p * dq}) = c'(q).$$
$$p(q)(1 + \frac{1}{\epsilon(q)}) = c'(q).$$
$$p(q) + \frac{p(q)}{\epsilon(q)}) = c'(q).$$
$$\frac{1}{\epsilon(q)}) = \frac{c'(q) - p(q)}{p(q)}.$$
$$-\frac{1}{\epsilon(q)}) = \frac{p(q) - c'(q)}{p(q)}.$$

Where $\epsilon(q) = \frac{p}{q}\frac{dq}{dp}$. So on the RHS of the last equation is the monopolist's "markup" or $(P - MC)/P$. So "markup is inversely proportional to the elasticity of demand." If demand is very elastic (high ϵ), the markup is low so the monopoly price is close to the competitive price.

- See G-13.6 for a plot of the monopolist's situation. Notice the monopolist will only set q on the elastic portion of the demand curve (above where $MR = 0$). Geometrically, the point B, the point on the demand curve corresponding to the optimal price and quantity is the midpoint of the competitive point (A) and the choke point (D), but this is only if demand is linear and costs are linear.

- Comparative Statics in the Classic Monopoly. Consider a monopolist's profit function with constant MC of c:

$$\pi(q, c) = qp(q) - cq - fixed\ costs.$$

Define the optimal quantity,

$$q(c) = arg\ max_{q \geq 0}\ \pi(q, c).$$

Note:

$$\frac{\partial \pi(q, c)}{\partial q} = p(q) + qp'(q) - c.$$

$$\frac{\partial^2 \pi(q, c)}{\partial q^2} = p'(q) + qp''(q) + p'(q) = 2p'(q) + qp''(q).$$

$$\frac{\partial^2 \pi(q, c)}{\partial q \partial c} = -1.$$

Then, by the envelope theorem, $q(c)$ must satisfy:

$$\frac{\partial \pi(q(c), c)}{\partial q} = 0.$$

Completely differentiating:

$$\frac{\partial^2 \pi(q(c), c)}{\partial q^2} \frac{dq}{dc} + \frac{\partial^2 \pi(q(c), c)}{\partial q \partial c} = 0.$$

Solve for dq/dc:

$$\frac{\partial^2 \pi(q(c), c)}{\partial q^2} \frac{dq}{dc} = -\frac{\partial^2 \pi(q(c), c)}{\partial q \partial c}.$$

$$\frac{dq}{dc} = -\frac{\partial^2 \pi(q(c), c)}{\partial q \partial c} \Bigg/ \frac{\partial^2 \pi(q(c), c)}{\partial q^2}.$$

Substitute in from above:

$$\frac{dq}{dc} = -(-1) \Big/ 2p'(q) + qp''(q).$$

$$\frac{dq}{dc} = \frac{1}{2p'(q) + qp''(q)}.$$

Therefore:

$$\frac{dp(q)}{dc} = \frac{dp(q)}{dq} \frac{dq}{dc} = \frac{p'(q)}{2p'(q) + qp''(q)}.$$

$$\frac{dp(q)}{dc} = \frac{1}{2 + qp''(q)/p'(q)}.$$

And finally note that when demand is linear, $p''(q) = 0$, so:

$$\frac{dp(q)}{dc} = \frac{1}{2 + q * 0/p'(q)} = \frac{1}{2}.$$

Which is intuitive from the graph. The increase in price resulting from a 10% increase in the marginal cost is 5% (only under linear demand).

- See G-13.7 for a graph of the government regulation solution to this problem. Note that in order to eliminate the DWL all together, the government must regulate the monopolist to set the price equal to the competitive price. However, if there are fixed costs, this will mean the monopolist will be making loses. Thus, the regulator sets:

$$q^{reg} = max\{q : qp(q) - c(q) \geq 0\}.$$

Assuming there is a fixed cost, there will still be a DWL, but it will be much smaller (see graph). It's the best we can do.

Review for Midterm

13.3 Preference Assumptions

- Completeness: $x \succeq y$ and/or $y \succeq x$.
- Transitivity: $x \succeq y, y \succeq z \Rightarrow x \succeq z$.
- Continuity: $\{x_i\}_1^n \to x, \{y_i\}_1^n \to y, x_i \succeq y_i \Rightarrow x \succeq y$.
- Strongly Monotone: $y \geq x$ and $y \neq x \Rightarrow y \succ x$.
- Monotone: $y >> x \Rightarrow y \succ x$.
- Local Non-Satiation: $\forall\, x \in X, \epsilon > 0, \exists\, y \in X \;\ni ||y - x|| < \epsilon$ and $y \succ x$.
- Convexity: $y \succeq x$, $z \succeq x$, $y \neq z \Rightarrow \alpha y + (1 - \alpha) z \succeq x$.

13.4 Properties of Functions

- $x(p, w)$ (Demand Correspondence from UMP)
  - (1) Hod(0) in (p, w).
  - (2) Walras Law: $w = px \;\forall\, x \in x(p, w)$.
  - (3) $\succeq$ convex $\Rightarrow x(p, w)$ is a convex set.
  - (4) $\succeq$ strictly convex $\Rightarrow x(p, w)$ is single valued.
- $v(p, w)$ (Indirect Utility Function)
  - (1) Hod(0) in (p, w).
  - (2) Strictly increasing in w and weakly decreasing in p_l.
  - (3) Quasiconvex in (p, w).
  - (4) Continuous in (p, w).
- $h(p, u)$ (Hicksian Demand Correspondence)
  - (1) Hod(0) in p.
  - (2) No Excess Utility. $u(x) = x \;\forall\, x \in h(p, u)$ or $u(h(p, \overline{u})) = \overline{u}$.
  - (3) $\succeq$ convex $\Rightarrow h(p, u)$ is a convex set.
  - (4) $\succeq$ strictly convex $\Rightarrow h(p, u)$ is single valued.
  - (*) $D_p h(p, u) = D_p^2 e(p, u)$: negative semidefinite, symmetric, and $D_p h(p, u) p = 0$.
- $e(p, u)$ (Expenditure Function)
  - (1) Hod(1) in p.
  - (2) Strictly increasing in u and weakly increasing in p_l.

- (3) Concave in p.
- (4) Continuous in p and u.

- $y(p)$ (Supply Correspondence)
  - (1) Upper hemi-continuous in p.
  - (2) If Y is convex, $Y = \{y \in \Re^L : p \cdot y \leq \pi(p) \; \forall \; p >> 0\}$.
  - (3) Hod(0) in p. (double input and output prices).
  - (4) If Y is convex, $y(p)$ is a convex set for all p. If Y is strictly convex, $y(p)$ is single valued.
  - (*) $D_p y(p) = D_p^2 \pi(p)$: positive semidefinite, symmetric, and $D_p y(p) p = 0$.

- $\pi(p)$ (Profit Function)
  - (1) Continuous in p.
  - (2) Hod(1) in p.
  - (3) Convex.

- $z(w, q)$ (Input Demand Function)
  - (1) Hod(0) in w.
  - (2) No excess production.
  - (3) Upper hemi-continuous in (w, q).
  - (4) If $f(z)$ is quasi-concave, $z(w, q)$ is a convex set, and if $f(z)$ is strictly quasi-concave, $z(w, q)$ is single valued.

- $C(w, q)$ (Cost Function)
  - (1) Hod(1) in w
  - (2) Strictly increasing in q.
  - (3) Continuous in (w, q).
  - (4) Concave in w.

- $x(p, w)$ (Proper Demand Function - From Integrability)
  - (1) Continuously differentiable.
  - (2) Satisfies Walras Law.
  - (3) $D_p h(p, u)$ symmetric and negative semidefinite.

- $x(p, w)$ (Proper Cost Function - From Integrability)
  - (1) $C(w, 0) = 0$.
  - (2) Continous.
  - (3) Strictly increasing q. Weakly increasing in w.
  - (4) Hod(1) in w.
  - (5) Concave in w.

13.5 Properties of the Production Set, Y

- (1) Non-empty, (2) Closed, (3) No free lunch, (4) Inaction, (5) Free disposal, (6) Irreversibility.
- (7) Nonincreasing RTS: $y \in Y$ and $\alpha \in [0, 1]$ then $\alpha y \in Y$.
- (8) Nondecreasing RTS: $y \in Y$ and $\alpha > 1$, then $\alpha y \in Y$.
- (9) Constant RTS: $y \in Y$ and $\alpha \geq 0$, then $\alpha y \in Y$.
- (10) Additivity, (11) Convexity, (12) Convex Cone = Convexity + CRS.

13.6 Duality

- Key Relationships:
  - (1) $e(p, v(p, w)) = w$.
  - (2) $v(p, e(p, u)) = u$.
  - (3) $h(p, u) = x(p, e(p, u))$.
  - (4) $x(p, w) = h(p, v(p, w))$.
- Shephard's Lemma: $h(p, u) = \nabla_p e(p, u)$.
- Roy's Identity:

$$x_l(p, w) = -\frac{\partial v(p, w)/\partial p_l}{\partial v(p, w)/\partial w}, \; for \; l = 1 \ldots L.$$

- Slutsky's Equation:

$$\frac{\partial h_l(p, u)}{\partial p_k} = \frac{\partial x_l(p, w)}{\partial p_k} + \frac{\partial x_l(p, w)}{\partial w} * x_k(p, w), \; for \; l, k = 1 \ldots L.$$

- Hotelling's Lemma:

$$\nabla_p \pi(p) = y(p).$$

- Shephard's Lemma:

$$\nabla_w C(w, q) = z(w, q).$$

13.7 Utility and Profit Maximization Problems

- Utility Maximization.
  - $v(p, w) = max \; u(x)$ s.t. $px \leq w$.
  - $x(p, w) = arg \; max \; u(x)$ s.t. $px \leq w$.
  - $D^2 u(x^*)$ negative semidefinite.
  - Lagrangian is the marginal utility of wealth.

- Expenditure Minimization.
  - $e(p, u) = min\ px$ s.t. $u(x) \geq u$.
  - $h(p, u) = arg\ min\ px$ s.t. $u(x) \geq u$.
  - $D_p h(p, u) = D_p^2 e(p, u)$: negative semidefinite, symmetric.
  - Lagrangian is the increase in wealth required to increase utility by one unit: $\lambda^{EMP} = 1/\lambda^{UMP}$.
- Profit Maximization (Production Set Notation).
  - $\pi(p) = max\ py$ s.t. $F(y) \leq 0$.
  - $y(p) = arg\ max\ py$ s.t. $F(y) \leq 0$.
- Profit Maximization (Production Function Notation).
  - $max\ pf(z) - wz$.
  - Must include complementary slackness ($z_i \geq 0$).
  - $D_p y(p) = D_p^2 \pi(p)$: positive semidefinite, symmetric, and $D_p y(p) p = 0$.
- Cost Minimization.
  - $C(w, q) = min\ wz$ s.t. $f(z) \geq q$.
  - $z(w, q) = arg\ min\ wz$ s.t. $f(z) \geq q$.
  - $D_w z(w, q) = D_w^2 C(w, q)$: negative semidefinite, symmetric.

13.8 Key Definitions and Propositions

- Concave: $u(\alpha x + (1 - \alpha) y) \geq \alpha u(x) + (1 - \alpha) u(y)$.
- Quasiconcave: $u(\alpha x + (1 - \alpha) y) \geq min\{u(x), u(y)\}$. (Utility Functions)
- Quasiconvex: $u(\alpha x + (1 - \alpha) y) \leq max\{u(x), u(y)\}$. (Indirect Utility Functions) Alternative formulation:

$$v(p, w) \text{ is Quasiconvex if } \{p : v(p, w) \leq \overline{v}\} \text{ is convex.}$$

- Euler's Formula.
  - In General: If $F(x)$ is hod(r) in x then

$$\sum_i \frac{\partial F(x)}{\partial x_i} x_i = rF(x).$$

– Walrasian demand. Since $x(p,w)$ is hod(0) in (p,w),

$$\sum_{k=1}^{L} p_k \frac{\partial x_l(p,w)}{\partial p_k} + w\frac{\partial x_l(p,w)}{\partial w} = 0 \; for \; l = 1 \dots L.$$

Found by differentiating $x(\alpha p, \alpha w) = x(p,w)$ wrt α and let $\alpha = 1$.

- Cournot Aggregation (diff W.L. wrt p):

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0 \; for \; k = 1 \dots L.$$

- Engle Aggregation (diff W.L. wrt w):

$$\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} = 1.$$

- Compensated Law of Demand:

$$(p'' - p') \cdot [h(p'', u) - h(p', u)] \leq 0.$$

- Equivalent Variation (Old Prices):

$$EV(p^0, p^1, w) = e(p^0, u^1) - e(p^0, u^0) = \int_{p_1^1}^{p_1^0} h_1(p_1, p_{-1}, u^1) dp_1.$$

- Compensating Variation (New Prices):

$$CV(p^0, p^1, w) = e(p^1, u^1) - e(p^1, u^0) = \int_{p_1^1}^{p_1^0} h_1(p_1, p_{-1}, u^0) dp_1.$$

- Consumer Surplus:

$$CS(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} x(p_1, p_{-1}, w) dp_1.$$

- Weak Axiom of Revealed Preferences (WARP) – does not provide symmetry of $D_p h(p,u)$.

$$p^1 \cdot x(p^2, w^2) \leq w^1, x(p^1, w^1) \neq x(p^2, w^2) \Longrightarrow p^2 \cdot x(p^1, w^1) > w^2.$$

$$\Longleftrightarrow$$

$$(p^2 - p^1) \cdot [x(p^2, w^2) - x(p^1, w^1)] \leq 0, \;\; with \; w^2 = p^2 \cdot x(p^1, w^1).$$

- Strong Axiom of Revealted Preferences (SARP) – adds transitivity and yields symmetry.

- Gorman Form Utility:
$$v_i(p, w_i) = a_i(p) + b(p)w_i.$$

- Uncompensated Law of Demand:
$$(p'' - p') \cdot [x_i(p'', w_i) - x_i(p', w_i)] \leq 0.$$

- Law of Supply:
$$(p'' - p')(y'' - y') \geq 0.$$

- Law of Input Demands:
$$(w'' - w')(z'' - z') \leq 0.$$

- Law of aggregate supply:
$$(p'' - p') \cdot (y(p'') - y(p')) \geq 0.$$

13.9 Notes from Problem Sets

- Look for perfect competition in problems to utilize $P = MC$.
- EV is an indirect utility function (old prices). If hicksian demands do NOT depend on u, then $EV = CV$.
- Leontief Production (Fixed Coefficients): $f(z_1, z_2) = q = min\{z_1, z_2\}$.
- CES utility: $u(x_1, x_2) = (\alpha_1 x_1^\rho + \alpha_2 x_2^\rho)^{1/\rho}$.
- Cobb-Douglas utility: $u(x_1, x_2) = x_1^\alpha x_2^{1-\alpha}$ yields demands: $x_1 = \alpha(w/p_1)$, $x_2 = (1 - \alpha)(w/p_2)$.
- Possibility of Inaction + Convexity = Nonincreasing RTS.
- Gross substitutes: $\partial x_i / \partial p_j > 0$, Net substitutes: $\partial h_i / \partial p_j > 0$.
- Preferences are strictly convex if $u(\cdot)$ is strictly quasiconcave and if $u(\cdot)$ is quasiconcave, SOC satisfied.
- $ln(x)$ is strictly concave.
- If the MRS of two utility functions are the same, then they represent the same preferences.
- If $u(x)$ is hod 1 in x, $x(p, w)$ is hod(1) in w and the income elasticity of demand is 1.
- $e(p, u)$ is concave if $\nabla_p^2 e(p, u)$ is negative semidefinite. For $L = 2$, $e_{11} \leq 0$, $e_{22} \leq 0$, $|H| \geq 0$.
- If $u(x)$ is quasilinear with respect to good 1, $u(x) = x_1 + \phi(x_2, \ldots, x_L)$.

- A C^1 function, $x(p,w)$, which satisfies walras law, and $D_p h(p,u)$ is symmetric and negative semidefinite IS the demand function generated by some increasing quasiconcave utility function.

- If $u(x)$ is homothetic, $e(p,u) = \phi(u)\psi(p) \Longrightarrow h(p,u) = \phi(u)\tilde{\psi}(p)$.

- Notes from PS 6:
$$\frac{\partial C(w,q)}{\partial w}\frac{w}{C(w,q)} = \frac{\partial ln\ C(w,q)}{\partial ln\ w}.$$
If $f(z)$ is $hod(r)$, then:
$$\frac{\partial ln\ C(w,q)}{\partial ln\ q} = \frac{1}{r}.$$

- Notes from PS 5:
Y CRS $\Longleftrightarrow f(\cdot)$ is hod(1).
Y convex $\Longleftrightarrow f(z)$ is concave.
$f(z)$ is hod(1) $\Longleftrightarrow c(w,q)$ and $z(w,q)$ are hod(1) in q.

- Normal good: h_l steeper than x_l. Inferior good: x_l steeper than h_l.

13.10 Two Problems

$u(x)$ **is hod(1)** $\Longrightarrow x(p,w)$ **is hod(1) in** w

Suppose $x^* \in x(p,w)$.
$$\Longrightarrow px^* \le w.$$
$$\Longrightarrow p\alpha x^* \le \alpha w.$$
So αx^* is affordable at αw. FEASIBLE.

Now choose any $y \in X$ such that $py \le \alpha w$. So y is affordable at $(p, \alpha w)$.
$$\Longrightarrow py \le \alpha w.$$
$$\Longrightarrow p\frac{y}{\alpha} \le w.$$
So $\frac{y}{\alpha}$ is affordable at (p,w). But since x^* is optimal at (p,w):
$$\Longrightarrow u(\frac{y}{\alpha}) \le u(x^*).$$
$$\Longrightarrow \alpha u(\frac{y}{\alpha}) \le \alpha u(x^*).$$
$$\Longrightarrow u(y) \le u(\alpha x^*).$$
So $\alpha x^* = x(p, \alpha w)$. OPTIMAL.

Y satisfies CRS $\Longrightarrow$ $f(\cdot)$ is hod(1)

- ($\Longrightarrow$)
  Start with an intial bundle $(-z, f(z)) \in Y$. By CRS,
  $$(-\alpha z, \alpha f(z)) \in Y.$$
  But $f(\alpha z) \geq \alpha f(z)$ because $f(\alpha z)$ is the maximum production at αz. So,
  $$(-\alpha z, f(\alpha z)) \in Y.$$
  Again by CRS:
  $$(-z, \frac{1}{\alpha} f(\alpha z)) \in Y.$$
  But $f(z) \geq \dfrac{1}{\alpha} f(\alpha z)$ because $f(z)$ is the maximum production at z. So,
  $$\alpha f(z) \geq f(\alpha z).$$
  Since $f(\alpha z) \geq \alpha f(z)$ and $\alpha f(z) \geq f(\alpha z)$,
  $$\alpha f(z) = f(\alpha z),$$
  and f is hod(1).

- ($\Longleftarrow$)
  Start with an initial bundle $(-z, q) \in Y$. By definition:
  $$f(z) \geq q.$$
  Thus,
  $$\alpha q \leq \alpha f(z) = f(\alpha z).$$
  This implies:
  $$(-\alpha z, f(\alpha z)) \in Y.$$
  Since $\alpha q \leq f(\alpha z)$, by free disposal:
  $$(-\alpha z, \alpha q) \in Y.$$
  So Y is CRS.

14 Lecture 14: October 14, 2004

14.1 Price Discrimination

Nonuniform/Linear Pricing - 3^{rd} Degree Price Discimination

- Let $p_1(q_1)$ be the inverse demand function for low-value consumers and $p_2(q_2)$ be the inverse demand for high-value consumers. High-valued consumers have more demand for the good. Accume $MC = c$.

- Monopolist's Problem:

$$max_{\{q_1,q_2\}}\left\{\underbrace{q_1 * p_1(q_1) + q_2 * p_2(q_2)}_{Linear\ Pricing} - c(q_1 + q_2) - fixed\ costs\right\}.$$

- FOCs:

$$p_1(q_1) + q_1 * p_1'(q_1) = c.$$
$$p_2(q_2) + q_2 * p_2'(q_2) = c.$$

- Rewrite as before as a markup:

$$\frac{p_1(q_1) - c}{p_1(q_1)} = -\frac{1}{\epsilon_1(q_1)}.$$

$$\frac{p_2(q_2) - c}{p_2(q_2)} = -\frac{1}{\epsilon_2(q_2)}.$$

  So the more inelastic is the demand (ϵ_i smaller), the larger the markup. See G-14.1 for what this looks like. We basically have two separate markets to work in and set price and quantity according to the classic monopoly setup in each market. The requirements for this setup are:

  - (1) Ability to identify the markets, no resale.
  - (2) Need a self-selecting tariff such that the pricing scheme leads to voluntary self-selection into the different groups.

- One might consider selling drugs in Canada and the US as a good example.

The Idea Behind Nonlinear Pricing (2 Part Tariff)

- Assume all consumers are identical and they pay a fixed amount E for the privilege to buy at price p. So total cost to consumer is:

$$E + p(q) * q.$$

Costco situation. See G-14.2. Optimal monopolist strategy is to set $p = c$ and charge a fixed fee $E = CS$, the consumer surplus. In this case, all demand above cost is served so there is no DWL so in a way this is better than the simple monopoly outcome.

- Now assume there are two types of consumers ("Disneyland Dilema" QJE 1971). See G-14.3. If price equals marginal cost, then E is the same area as above. However this is not optimal. Optimal scheme is to set $p = c + \Delta > c$. The new entry fee at the higher price is shaded yellow and the overall gain to the monopolist is shaded green. So the additional profit is $q_2 * \Delta$ and the loss is coming from the resulting smaller entry fee.
- For a small enough Δ, the monopolist is strictly better off charging a price above c. The distortion created by the 2-part tariff makes the monopolist better off. Note there is a DWL associated with this type of pricing scheme.

Anonymous Nonlinear Pricing (2^{nd} Degree Price Discrimination)

- Assume 2 commodities, x, the monopolized good, and y, everything else. $p_y = 1$.
- There are 2 types of consumers with utility:

$$u_1(x_1, y_1) = u_1(x_1) + y_1,$$

$$u_2(x_2, y_2) = u_2(x_2) + y_2.$$

- A menu is offered to each consumer. Choose either:

$$(r_1, x_1), \;\; or \;\; (r_2, x_2).$$

Where r_i is the payment from the consumer including the entry fee. It is really the revenue to the monopolist. And x_i is the total quantity of the good received.

- We need an assumption that is displayed in G-14.4. The Single Cross Property. Assume:

$$u_2'(x) > u_1'(x) \; \forall \; x \geq 0, \; and, \; u_1(0) = u_2(0) = 0.$$

This implies:

$$u_2(x) > u_1(x) \; \forall \; x > 0.$$

The slopes of consumer 2's utility function is greater than consumer 1's and if they both start at 0, consumer 2 must be extracting more utility than consumer 1 at all levels of consumption.

- Monopolist's problem:

$$max_{r_1, r_2, x_1, x_2}\{r_1 + r_2 - c(x_1 + x_2)\}.$$

Subject to:

- (1) $u_1(x_1) - r_1 \geq 0$.
- (2) $u_2(x_2) - r_2 \geq 0$.
- (3) $u_1(x_1) - r_1 \geq u_1(x_2) - r_2$.
- (4) $u_2(x_2) - r_2 \geq u_2(x_1) - r_1$.

Constraints (1) and (2) are "Individual Rationality Constraints" or "Participation Constraints". (3) and (4) are "Incentive Compatibility Constraints" or "Self-Selection Constraints." So the menu options must be such that each consumer is willing to buy the good and it also must be the case that they prefer to buy the good they are matched with instead of the other. Ie, consumer 1 must want to buy good 1 and she must want to buy good 1 more than she wants to buy good 2. Self-Selection.

- Argument that (2) and (3) will NOT bind but (1) and (4) will:
  - (1) Monopolist wants to make r_1 and r_2 as large as possible. So only (1) and (3) limit that value of r_1. So at least (1) or (3) must bind or the problem would be unbounded. The same is true for (2) and (4), one and/or the other must bind.
  - (2) From the single cross property (SCP), $u_2(x) > u_1(x) \; \forall \; x > 0$. So:

    $$(1) : u_1(x_1) \geq r_1 \; PLUS \; u_2(x) > u_1(x) \; YIELDS \; u_2(x) > u_1(x) \geq r_1.$$

    But this implies for the RHS of (4):

    $$(4) : u_2(x_2) - r_2 > \underbrace{u_2(x_1) - r_1}_{>0}.$$

    Which means the LHS of (4), which is also (2), does NOT bind:

    $$(2) : u_2(x_2) - r_2 > 0.$$

    So if (2) does not bind, then (4) BINDS.
  - (3) If consumer 2 is the high value consumer, then $r_2 > r_1$. Rearrange (4) – recall that (4) is binding:

    $$(4) : u_2(x_2) - u_2(x_1) = \underbrace{r_2 - r_1}_{>0}.$$

    So the LHS of this last equation impiles $x_2 > x_1$. By the SCP:

    $$u_2(x_2) - u_2(x_1) > u_1(x_2) - u_1(x_1).$$

    So,

    $$r_2 - r_1 > u_1(x_2) - u_1(x_1).$$

    Rearranging:

    $$u_1(x_1) - r_1 > u_1(x_2) - r_2.$$

    And this means that (3) does NOT bind. So (1) must bind.

- So by (1), $u_1(x_1) = r_1$, and by (4), $r_2 = u_2(x_2) - u_2(x_1) + u_1(x_1)$. Monopolist's problem reduces to:

$$max_{x_1,x_2}\Big\{u_1(x_1) + u_2(x_2) - u_2(x_1) + u_1(x_1) - c(x_1 + x_2).\Big\}.$$

- FOCs:

$$u_1'(x_1) = c + \underbrace{u_2'(x_1) + u_1'(x_1)}_{>0\ by\ SCP} > c.$$

And,

$$u_2'(x_2) = c.$$

- So the high value consumer will buy the efficient quantity while the low value consumer will buy less at a higher price.
- This whole analysis assumes the monopolist sells to both types of consumers. One should check to see if he could do better by just selling to type 2. Under this setting:

$$r_2 = u(x_2),\ where\ u'(x_2) = c.$$

Nonlinear/Nonuniform Pricing - 1^{st} Degree Price Discimination

- The monopolist's problem:

$$max_{r_1,r_2,x_1,x_2}\{r_1 + r_2 - c(x_1 + x_2)\}.$$

There is NO need for self selection here. We can identify perfectly what price to charge to whom. Constraints (rationality):

$$u_1(x_1) \geq r_1.$$

$$u_2(x_2) \geq r_2.$$

- Solution:

$$x_1 \ni u_1'(x_1) = c \Longrightarrow \text{Efficient Quantity.}$$

$$x_2 \ni u_2'(x_2) = c \Longrightarrow \text{Efficient Quantity.}$$

$$r_1 = u_1(x_1) \Longrightarrow \text{Extract All Surplus.}$$

$$r_2 = u_2(x_2) \Longrightarrow \text{Extract All Surplus.}$$

We're back to the situation in G-14.2.

15 Lecture 15: October 19, 2004

15.1 Game Theory

Static Games of Complete Information

- **Definition:** An n-player static game of complete information consists of an n-tuple of strategy sets and an n-tuple of payoff functions denoted:

$$G = \{S_1, \ldots, S_n; u_1, \ldots, u_n\}.$$

Where S_i is the strategy set for player i with $s_i \in S_i$ and u_i is the payoff for player i (utility, profits, etc) where:

$$u_i = u_i(s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n).$$

- Assume players move independently and simultaneously. Since these are static 1 period games, all moves occur at the same moment in time.

- Consider the "prisoners dilemma" game in G-15.1. The unique Nash Equilibrium (NE) is for both prisoners is to confess: $S_1 = S_2 = \{Confess, Confess\}$.

- Consider the "battle of the sexes" games in G-15.2. There are two NE in this game: $\{Boxing, Boxing\}$ and $\{Ballet, Ballet\}$ because the most important thing is being together.

- **Definition:** A Nash Equilibrium (NE) of a game G in pure strategies consists of a strategy for every player with the property that no player can improve her payoff by unilaterally deviating. So:

$$(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \ni$$

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \; \forall s_i \in S_i.$$

Equivalently, a NE is a mutual best response. That is, for every player i, s_i^* is a solution to:

$$s_i^* \in arg\ max_{s_i \in S_i} \ \{u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*)\}.$$

- **Definition:** A Strict Nash Equilibrium (SNE) of a game G in pure strategies consists of a strategy in which every player would be made worse off by unilaterally deviating. So:

$$(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \ni$$

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) > u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*) \; \forall s_i \in S_i, s_i \neq s_i^*.$$

- Cournot Model of Oligopoly. Consider a model with n firms, each firm with constant marginal cost, c_i. The aggregate inverse demand function is $P(Q)$. Each firm simultaneously and independently selects a strategy consisting of a quantity, $q_i \in [0, a]$ where

$P(a) = 0.$
Payoff Functions:

$$\pi_1(q_1, q_2) = q_1 P(q_1 + q_2) - c_1 q_1.$$
$$\pi_2(q_1, q_2) = q_2 P(q_1 + q_2) - c_2 q_2.$$

Strategies:

$$S_1 = S_2 = [0, a].$$

Assume $c_1 = c_2 = c$ and linear demand, $P(Q) = a - Q$. Solution: (q_1^*, q_2^*) is a NE iff q_1^* solves:

$$max_{q_1}\ \{q_1[P(q_1 + q_2^*) - c]\} = max_{q_1}\ q_1[(a - q_1 - q_2^*) - c] = max_{q_1}\ aq_1 - q_1^2 - q_1 q_2^* - cq_1,$$

and, q_2^* solves:

$$max_{q_2}\ \{q_2[P(q_1^* + q_2) - c]\}.$$

FOCs:

$$a - 2q_1 - q_2 - c = 0. \quad (1)$$
$$a - 2q_2 - q_1 - c = 0. \quad (2)$$

Subtract (2) from (1):

$$-2q_1 + q_1 - q_2 + 2q_2 = 0 \Longrightarrow -q_1 + q_2 = 0 \Longrightarrow q_1^* = q_2^*.$$

Substitute into (1):

$$a - 2q_1 - q_1 - c = 0. \quad (1')$$
$$-3q_1 = c - a.$$
$$q_1^* = \frac{a - c}{3}.$$

By symmetry:

$$q_2^* = \frac{a - c}{3}.$$

16 Lecture 16: October 21, 2004

16.1 Game Theory

More on the Cournot Game (1838) - Competition in Quantities

- See G-16.1 for a graph of firm 1's residual demand given that firm 2 has chosen to produce q_2^*. Once firm 2 has choosen, firm 1 acts as a monopolist on the residual demand and graphically, sets price at the midpoint between the choke price and the point the residual demand hits the marginal cost. This (*) point is:

$$(q, p) = (\frac{a - q_2 - c}{2}, \frac{a - q_2 + c}{2}).$$

  And this is firm 1's (and by symmetry, firm 2's) best response function:

$$R_1(q_2) = (a - q_2 - c)/2.$$

$$R_2(q_1) = (a - q_1 - c)/2.$$

  These functions describe firm i's best response to whatever firm j has chosen. See G-16.2. Where R_1 and R_2 cross is the NE (mutual best response). Note in this graph, we have set $c = 0$ and $a = 1$.

Bertrand Model of Oligopoly (1883) - Competition in Prices

- Consider n firms each with constant marginal cost c_i. Aggregate demand is $Q(p)$. Firms select prices $p_i \in [0, a]$ where $Q(a) = 0$.
- Payoff functions when $n = 2$:

$$\pi_1(p_1, p_2) = \begin{cases} Q(p_1)[p_1 - c_1] & p_1 < p_2 \\ \frac{1}{2}Q(p_1)[p_1 - c_1] & p_1 = p_2 \\ 0 & p_1 > p_2 \end{cases}$$

  And,

$$\pi_2(p_1, p_2) = \begin{cases} Q(p_2)[p_2 - c_2] & p_2 < p_1 \\ \frac{1}{2}Q(p_2)[p_2 - c_2] & p_2 = p_1 \\ 0 & p_2 > p_1 \end{cases}$$

- Strategy sets: $S_1 = S_2 = [0, a]$.
- Solution:
  - (1) First observe that at any NE, $p_1^* > c$ and $p_2^* > c$.
    Proof: Suppose $p_1^* < c$ and $p_1^* \leq p_2^*$. So firm 1 is earning strictly negative profits so it could deviate and raise prices and get at least zero profits by setting p_1^* above

c. If $p_1^* > p_2^*$, then firm 1 receives 0 profits and if $c < p_1^* \leq p_2^*$, p_1 earns positive profits.

 - (2) Second observe that at any NE, $p_1^* = p_2^*$.
 Proof: Suppose $c \leq p_1^* < p_2^*$. Firm 2 is earning zero profits and as long as $p_1^* > c$, firm 2 should set $p_2^* = p_1^* - \epsilon$ to barely undercut and capture the market. If $p_1^* = c$, then firm 1 should deviate and set $p_1^* = p_2^* - \epsilon$ to increase profits. So prices converge until we have $p_1^* = p_2^*$.
 - (3) Third observe that at any NE, $p_1^* = p_2^* = c$.
 Proof: The only remaining possibility given (1) and (2) is $p_1^* = p_2^* = p^* \geq c$. Each firm would earn profits of :

 $$m_1 = 1/2D(p^*)[p^* - c].$$

 But if $p^* > c$, each firm can profitably deviate by setting $p_i^* = p^* - \epsilon$ and capturing the entire market earning:

 $$m_2 = D(p^* - \epsilon)[p^* - \epsilon - c].$$

 For small enough ϵ, $m_2 > m_1$. Thus this uncutting continues until $p^* = c$ and no more profitable deviations exist.

- Note that this NE, $p_1^* = p_2^* = c$ with zero profits for each firm, is NOT a STRICT NE. Notice that if either firm raises their price above p^*, they continue to earn zero profits. Thus the NE is not strict. We only look at deviations of one player and see if they can be as well off at another strategy (even if it is NOT nash!) Of course one firm setting a price above marginal cost is not a best response, but it shows that the zero profit NE is not strict in that firm 1 could earn zero profits with another strategy (which is out of equilibrium).

Pollution Game

- Consumers choose between 3 cars: A, B, and C. The cars are identical except for their price and pollution emissions:

Model	Price	Emissions (e)
A	$15,000	100
B	$16,000	10
C	$17,000	0

- Consumers have utility:

$$u = v - p - E.$$

Where v is the reservation value of the car, p is the price paid, and $E = \sum_{i=1}^{n} e_i$, a monetary equivalent of the aggregate pollution caused.

- While we would like to think the socially optimal car would be choosen (all with car B if $n \approx 35$), in actuality, the NE strategy is to choose car A. Note:

$$\pi_i(A, s_{-i}) - \pi_i(B, s_{-i}) = 1000 - 90 = 910.$$

  So the consumer is made strictly better off by switching from car B to car A. Free rider problem. Solution is to make car A illegal to sell or to tax the difference between the social and private cost.

17 Lecture 17: October 26, 2004

17.1 Dominate Strategies

- **Definition:** A strategy s_i (strictly) dominates s_i' if for all possible strategy combintations of player i's opponents, s_i yields a (strictly) higher payoff than s_i' to player i.
- We can use this type of idea to find NE by "Iterated Elimination of Strictly Dominated Strategies" (IESDS). Note this only works for strictly dominated strategies such that the payoffs are strictly larger in the dominant strategy. More on this soon.
- **Proposition 1**. If IESDS yields a unique strategy n-tuple, this strategy n-tuple is the unique strict NE.
- **Proposition 2**. Every NE survives IESDS.
- See G-17.1 for IESDS on a simple game. Note the resulting NE is unique and strict. The deviator would be strictly worse off by unilaterally deviating.
- Footnotes to these two propositions:
  - (1) Not every game can be solved using IESDS (See Battle of the Sexes).
  - (2) Sometimes, proposition 2 may be helpful even if IESDS does not yield a unique NE. See G-17.2 which reduces to battle of the sexes after 2 rounds of IESDS.
  - (3) We need "strictly" dominated in the statement of propositions. Iteration of weakly dominated strategies will not work. Consider the Bertrand game where $P = MC$ was a NE and profits were zero. If $c = 10$ and $p_i^* = 10$, then p_i^* is weakly dominated by $p_i' = 20$. If $p_j' = 25$, this yields positive profit for firm i, while if $p_j' = 15$, this again yields 0 profit. So p_i^* is weakly (but NOT strictly) dominated by p_i'.
- The main point to take from this is that IESDS is order-independent. No matter how we eliminate strategies, we end up with the same result. This would not apply when using weakly dominated strategies.
- Now back to the response graph for the cournot game shown in G-17.3. Recall $c = 0$ and $a = 1$. Notice the following:
  - (I) $q_1 > \frac{1}{2}$ is strictly dominated by $q_1 = \frac{1}{2}$ because $q_1 = \frac{1}{2}$ is the monopoly quantity. Thus we shade the region (I) to show these strategies are dominated.
  - (II) The same is true for firm 2 (by symmetry) so region (II) is shaded.
  - (III) Since $R_1(q_2)$ in the shaded region will not be optimal (since $q_2 \leq \frac{1}{2}$), $q_1 < \frac{1}{4}$ is strictly dominated by $q_1 = \frac{1}{4}$. So shade region (III).
  - (IV) Repeat III for firm 2.

– Notice the resulting picture is identical to the original, only scaled down. So we can repeat this process and in the end, we "zoom-in" to the NE at $R_1(q_2) = R_2(q_1) \Longrightarrow q_1 = q_2 = \frac{1}{3}$.

18 Lecture 18: October 28, 2004

18.1 Mixed Strategies

- Consider the game of "matching pennies" in G-18.1. Note there are NO pure strategy NE in this game.
- **Definition:** Let player i have K pure strategies available. Then a mixed strategy for player i is a probability distribution over those K strategies. The strategy space for player i is denoted:
$$S_i = (s_{i1}, \ldots, s_{iK}).$$
And the mixed strategy:
$$p_i = (p_{i1}, \ldots, p_{iK}).$$
Note $\sum_{k=1}^{K} p_{ik} = 1$ and $0 \leq p_{ik} \leq 1$.
- So back in our game. Suppose player I randomizes between H and T, playing H with probability q. The means player I must be indifferent between H and T (ie the expected payoffs must be the same). Suppose also that player II randomizes between H and T and plays H with probability r. Thus, the expected payoff to player I from playing H is
$$r(1) + (1-r)(-1) = 2r - 1.$$
From playing T:
$$r(-1) + (1-r)(1) = 1 - 2r.$$
Since the expected payoffs are equal: $2r - 1 = 1 - 2r \Rightarrow r = \frac{1}{2}$. A similar argument shows that $q = \frac{1}{2}$.
- Thus the unique NE in mixed strategies is:
$$(p_{11}, p_{12}) = (p_{21}, p_{22}) = (\frac{1}{2}, \frac{1}{2}).$$
- See G-18.2 for the diagram showing the response functions for the two players. Note that if player II is playing each strategy with probability 1/2 then player I is indifferent between playing each strategy with any probability. He could play all heads, or all tails, but will still end up with an expected payoff of zero. The mixed strategy NE is at the intersection (unique).
- **Theorem:** Nash Existance Theorem (1950). Every finite game has at least ONE NE (possibly in mixed strategies). Where a finite game is a game with a finite number of players and a finite number of strategies for each player.
- Fact: If, in a mixed strategy NE, player i places positive probability on each of two strategies, then player i must be indifferent between these two strategies (ie, they must yield the same expected payoff). Otherwise, the player should only play the strategy with the higher expected payoff.

- Note also that whenever we are dealing with a mixed strategy NE, the NE cannot be strict since by definition, there must exist strategies with equal expected payoffs.

- In the battle of the sexes game in G-18.3, the man goes to the boxing match with probability q and the woman goes with probability r. Thus the man's expected payoff from boxing and ballet implies:

$$\underbrace{r(2) + (1-r)(0)}_{Boxing} = \underbrace{r(0) + (1-r)(1)}_{Ballet} \Longrightarrow r = \frac{1}{3}.$$

  And the woman's:

$$\underbrace{q(1) + (1-q)(0)}_{Boxing} = \underbrace{q(0) + (1-q)(2)}_{Ballet} \Longrightarrow q = \frac{2}{3}.$$

  So there are three mixed strategies in this game (2 pure and one mixed):

$$(\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, \frac{1}{3}),$$

$$(1, 0), (1, 0),$$

$$(0, 1), (0, 1).$$

  See G-18.4 for the best response plot. Notice there are three interesections!

- Finally consider the graph in G-18.5 which shows the payoffs to player I and player II in the battle of the sexes. Note that the mixed strategy NE is pareto dominated by both of the pure strategies! This is NOT a general result but it shows that sometimes the breakdown of an ability to bargain over pure strategies leads to a mixed strategy which yields a lower expected payoff for all involved.

19 Lecture 19: November 2, 2004

19.1 Fixed Point Theorems

- **Theorem** Brouwer's Fixed Point Theorem. Suppose X is a non-empty, compact, and convex set in $\Re^n$. Suppose the function $f : X \mapsto X$ is continuous. Then there exists a fixed point of f, ie a point $x \in X$ such that $f(x) = x$.

- **Theorem** Kakutani's Fixed Point Theorem. Suppose X is a non-empty, compact, and convex set in $\Re^n$. Suppose the correspondence $F : X \mapsto X$ is non-empty and convex-valued and $F(\cdot)$ has a closed graph. Then there exists a fixed point of F, ie a point $x \in X$ such that $x \in F(x)$.

- See G-19.1 for a picture of Brouwers in $\Re^1$. Note since the left end point must be above or on the 45 and the right end point must be below or on the 45, we must have a crossing (possibly many). For higher dimensions, this does not generalize but the theorem still holds.

- See G-19.2 for counter examples to Brouwers. If X is open, not bounded, not convex, or empty, the theorem may fail. Also f cannot be discontinuous and f must map X into itself or again, the theorem may fail. In the game theoretic sense, X will be the strategy set of all players and F will be the best response correspondence. We look to show that in all finite games, there will exists at least one NE (possibly in mixed strategies).

- For Kakutani's the only difference is that F is now a correspondence (as best response functions usually are). See G-19.3 for counterexamples for when F is not convex valued and when F does not have a closed graph. Note F being convex valued means the range of F is convex, so if x maps to multiple $F(x)$'s, then these $F(x)$'s must form a convex set. Having a closed graph means that the graph of F contains its limit points. Thus there cannot be any open circles in the graph. You could have a discontinuity though with a non-closed graph, but it must be that points are filled in on each end of the jump.

- So define X as the set of all mixed strategies for all players in a finite game. Suppose there are N players (finite) and each player has finitely many pure strategies. Suppose there are K_i pure strategies available to player i. Thus:

$$X = \underbrace{[0,1] * \cdots * [1,0]}_{K_1} * \underbrace{[0,1] * \cdots * [1,0]}_{K_2} * \cdots\cdots * \underbrace{[0,1] * \cdots * [1,0]}_{K_N}.$$

  Thus $X \subset \Re^n$ where,

$$n = \sum_{i=1}^{N} K_i,$$

  as required in the above theorems.

- Now consider the best response correspondence, F. If σ_i is a mixed strategy for player i, then:
$$F : \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix} \mapsto \begin{pmatrix} BR_1(\sigma_2, \sigma_3) \\ BR_2(\sigma_1, \sigma_3) \\ BR_3(\sigma_1, \sigma_2) \end{pmatrix}.$$
So a fixed point will exists, ie
$$\sigma_1 \in BR_1(\sigma_2, \sigma_3),$$
$$\sigma_2 \in BR_2(\sigma_1, \sigma_3),$$
$$\sigma_3 \in BR_3(\sigma_1, \sigma_2),$$
if the conditions of the Kakutani's theorem hold.

- Verifying Kakutani's properties for X and F defined above.
  - Is X nonempty? Yes, trivially.
  - Is X closed? Yes, sets of probabilities are closed.
  - Is X bounded? Yes, $x_i \in [0, 1] \ \forall \ x_i \in X$.
  - Is X convex? Yes, any convex combination of two probability vectors is also in X.
  - Is $X \subset \Re^n$? Yes, shown above using the fact that we have a finite game.
  - Does $F : X \mapsto X$? Yes, a best response is also a mixed strategy.
  - Is F nonempty valued? Yes, via weistrass (X is compact).
  - Is F convex valued? Yes, mixed strategies allow you to convexify. Given two strategies that yield the same expected payoff, a convex combination of those strategies will also yield the same expected payoff.
  - Does F have a closed graph. Suppose $(x^n, y^n) \to (x, y)$ with $y^n \in BR_i(x^n) \ \forall \ n$, but $y \notin BR_i(x)$. So the limit point is not in the best response correspondence. Then there exists $\epsilon > 0$ and $y' \neq y$ such that:
  $$u_i(y', x) > u_i(y, x) + \epsilon.$$
  But this contradicts:
  $$u_i(y', x^n) \leq u_i(y^n, x^n), \ \forall \ n.$$
  And this follows from the fact that y^n is a best response to x^n for all n. Thus the graph of of F contains its limit points which means that F has a closed graph.

- Thus, Kakutani holds and there exists a vector of mixed strategies with the property that all players are playing their best response to all other player's strategies. In other words, the fixed point which is guaranteed by the Kakutani Theorem is a Nash Equilibrium.

20 Lecture 20: November 4, 2004

20.1 Footnotes on Fixed Point Theorems

- One of the requirements for the FP theorems to hold was we had to have a finite game with finitely many strategies. Is this always the case? Usually, but there are at least two important exceptions:
  - (1) Cournot Game with n-players. Assume, instead of linear demand and constant MC, general demand and cost functions. Also assume:

    $$Max_q \Big\{ qP(Q+q) - C(q) \Big\},$$

    has a unique solution for all $Q > 0$, where Q is the quantity produced by the firm's opponents. This means the best response relation is a function and not a correspondence. In this case, we can invoke the Brouwer Fixed Point theorem with:

    $$X = \{\text{Pure Strategy Combinations of all Players}\}.$$

    If we included mixed strategies, X would again be infinite, but in this case, we have just N strategies.
  - (2) Bertrand game with unequal marginal cost. Recall that under equal marginal cost, we happen to have one unique NE. With $N = 2$ and $c_1 < c_2$ there is ALMOST a NE :

    $$P_2 = c_2, \quad P_1 = c_2 - \epsilon.$$

    However, firm 1 could profitably deviate by setting $P_1' = c_2 - \epsilon/2$. If we discretize the strategy space (say in cents), we get two NE:

    $$P_2 = c_2, \quad P_1 = c_2 - 0.01,$$

    $$P_2 = c_2 + 0.01, \quad P_1 = c_2.$$
- See G-20.1 for a picture showing why we usually see odd numbers of NE. This is clearly not ALWAYS the case but it is likely.

20.2 Hotelling Product Differentiation

- Consumers live uniformly along the interval $[0, 1]$. 2 Firms located at $x = 0$ and $x = 1$. Each produce the same good at the same cost, c. Consumers have a transportation cost of t per unit travelled to reach a firm. See G-20.2. Each consumer buys 0 or 1 unit with $u(0) = 0$ and $u(1) = v > 0$. Firm 1 charges p_1 and firm 2 charges p_2. A consumer located at x will receive:

  $$v - p_1 - tx \text{ if they buy from firm 1,}$$

and,

$$v - p_2 - t(1-x) \text{ if they buy from firm 2.}$$

- Equation of the marginal man (indifferent between going to firm 1 and 2):

$$v - p_1 - tx = v - p_2 - t(1-x).$$

$$p_2 - p_1 = tx - t + tx.$$

$$x^* = \frac{p_2 - p_1 + t}{2t} = \frac{1}{2} + \frac{p_2 - p_1}{2t}.$$

- Profits for the firms:

$$\pi_1(p_1, p_2) = (p_1 - c) * x^* = (p_1 - c)[\frac{1}{2} + \frac{p_2 - p_1}{2t}].$$

$$\pi_2(p_1, p_2) = (p_2 - c) * (1 - x^*) = (p_2 - c)[\frac{1}{2} - \frac{p_2 - p_1}{2t}].$$

- FOC for firm 1:

$$[\frac{1}{2} + \frac{p_2 - p_1}{2t}] + (p_1 - c)(-\frac{1}{2t}) = 0.$$

$$1 + \frac{p_2 - p_1}{t} = \frac{p_1 - c}{t}.$$

$$t + p_2 - p_1 = p_1 - c.$$

$$p_1 = \frac{t + p_2 + c}{2}.$$

- Symmetrically for firm 2:

$$p_2 = \frac{t + p_1 + c}{2}.$$

- Substituting yields:

$$p_1^* = p_2^* = t + c.$$

- This is very similar to the median voter theorem. The distance the firms are from each other is a type of horizontal product differentiation.

- Hotelling's 1929 Error. Technically, the problem with Hotelling's original model was the assumption of linear costs which could result in discontinuous jumps in the demand schedule as shown in the notes. Invoking quadratic cost curves (umbrellas) eliminates this possibility. However, the fundamental error is not just in the discontinuities. Having the linear costs meant that in equilibrium, two firms would choose to locate right next to each other on the line and split the market evenly. He referred to this as the "Principal of Minimal Differentiation." Invoking quadratic costs yielded the exact opposite conclusion: Two firms choose to locate at exactly the opposite ends of the line and thus the "Principal of Maximal Differentiation." The two forces at work here is first the position of the marginal man. Two firms located along the line at distinct points would both have an incentive to move towards their neighbor, thus shifting over the marginal man and gaining sales volume. However (the second force), moving closer to your neighbor also makes the products more substitutable (less differentiated) and thus drives down prices and profits. In an extreme case, with bertrand price competition, it is easy to see that two firms would want to be as far away from each other as possible because if they had to compete, prices would be driven down to marginal cost and both firms would make zero profit. Hotelling ignored the price competition idea in his original analysis and thus didn't see how a simple adjustment to the cost schedule could yield precisely the opposite results. The reason that the change to the cost schedule yields this idea is because moving away from your neighbor now results in higher profits because of the product differentiation effect. Before (with linear costs), this effect was too small and thus Hotelling's original conclusion. With quadratic costs, it is now beneficial to the firms to move away from each other, losing the sales volume from the shifting marginal man, but gaining positive profits from the increased product differentiation.

20.3 Dynamic Games of Complete Information

- See G-20.3 for an extensive form game of Battle of the Sexes. Note we can solve it by backwards induction and find that with sequential moves, there is only one NE.
- **Definition:** Subgame Perfect Equilibrium. In an n-player dynamic game of complete information, an n-tuple of strategies is said to form a Subgame Perfect Equilibrium (SPE) if the strategies constitute Nash Equilibria in every subgame.
- **Definition:** Information Set. A player does not know which node she is at. See G-20.4. Draw a circle around the nodes to signify they are in the same info set.
- Finally, it is important to note that the NE of dynamic games need not correspond in any way to the NE of static games. See G-20.5, sequential matching pennies. In the static game, there was one NE in mixed stratgies where both players randomized and the expected payoff to both players was 0. In the sequential game, the expected payoff to the player who moves second is $+1$ while the the payoff to the first mover is -1. The second players would NEVER randomize. Thus the first player can play any mixed strategy, any $(p, 1-p)$, and he will obtain the same payoff. Thus there are infinitely many NE in the dynamic game.

21 Lecture 21: November 9, 2004

- Consider the game in G-21.1 (Selten's Chain Store Paradox (1978). Note there are TWO NE of this game. The obvious one is: (Acquiesce if Enter, Entrant Enters). Clearly, under this strategy, no player has a profitable deviation. The other NE is: (Fight if Enter, Stay Out). In this case, it is still clear that no player has a unilateral deviation, but it relies on the fact that the incumbent will actually play Fight if in fact he is faced with an entrant.

- The second NE is NOT subgame perfect. It is not credible because if actually faced with an entrant, the incumbant would not fight, he would rather acquiesce.

- Note that the set of SPE is a subset of NE (G-21.2).

- Now suppose there are a sequence of N entrants that the incumbant faces, one after another. It might make intuitive sense to fight off the first few, develop a reputation, and then not have to face future entrants. However, since there is a final round, consider the game against the N^{th} entrant. It is exactly as in G-21.1 so the only SPE is (acquiesce, enter). Against the $(N-1)^{th}$ entrant, again, the incumbant has nothing to gain from fighting so he will again, acquiesce. Repeating this argument backwards, we see the only SPE is for all entrants to enter and the incumbant to acquiesce every time. Fighting is never a credible strategy.

- In a finite sequential game, the invokation of SPE has very strong implications on the set of equilibria.

- In infinite games or in games with an uncertain ending, this may be avoided though there is still the problem of even infinite games having an ending when the universe comes to an end.

22 Lecture 22: November 11, 2004

22.1 Stackelberg

- Consider the 2 firm game shown in G-22.1 where firm 1 chooses its quantity first and then firm 2, having observed firm 1's quantity, chooses its quantity. The payoffs are thus:

$$(q_1[P(q_1+q_2)-c], q_2[P(q_1+q_2)-c]).$$

Firm 1 is the stackelberg leader and firm 2 is the follower. Suppose demand is $P(Q) = a - Q$. Firm 2 solves:

$$Max_{q_2}\pi(q_2|q_1) \Rightarrow q_2^*(q_1) = R_2(q_1) = \frac{a-q_1-c}{2}.$$

And this is the cournot solution we had before. Now, firm 1 solves:

$$Max_{q_1}\pi(q_1) = q_1(a-q_1-q_2^*-c) \Rightarrow q_1^* = \frac{a-c}{2}.$$

So the set of NE strategies are:

$$q_1^* = \frac{a-c}{2}.$$

$$q_2^*(q_1) = \frac{a-q_1-c}{2}.$$

- Plugging q_1^* into $q_2^*(q_1)$ yields $q_2^* = \frac{a-c}{4}$. Note this means the equilibrium quantity and price is lower than in the cournot game for the stackelberg follower. The dynamics really cost the second player.

22.2 Bargaining

Alternating Offers - Finite Periods

- See the game in G-22.2. We use the following method of solving by backwards iteration:

At IV. B Accepts if: $\delta^2(1-P_3) \geq 0 \Rightarrow P_3 \leq 1 \Rightarrow P_3 = 1.$

At III. S Accepts if: $\delta P_2 \geq \delta^2 P_3 \Rightarrow P_2 \geq \delta \Rightarrow P_2 = \delta.$

At II. B Accepts if: $1-P_1 \geq \delta(1-\delta P_3) \Rightarrow P_1 \leq 1-\delta+\delta^2 P_3 \Rightarrow P_1 = 1-\delta+\delta^2.$

At I. S Offers: $P_1 = 1-\delta+\delta^2.$

Alternating Offers - Infinite Periods

- Note in the 1 period game, S offers $P = 1$; 3 periods, S offers $P = 1-\delta+\delta^2$; 5 periods, S offers $P = 1-\delta+\delta^2-\delta^3+\delta^4$; Infinite Odd Periods:

$$\text{S offers: } \; P = \sum_{i=0}^{\infty}(-\delta)^i = \frac{1}{1+\delta}.$$

- Note in the 2 period game, B offers $P = \delta$; 4 periods, B offers $P = \delta-\delta^2+\delta^3$; 6 periods, B offers $P = \delta-\delta^2+\delta^3-\delta^4+\delta^5$; Infinite Even Periods:

$$\text{B offers: } \; P = \sum_{i=0}^{\infty}\delta * (-\delta)^i = \frac{\delta}{1+\delta}.$$

- Conjecture based on the above analysis that the SPE is as follows:

$$\text{In every odd period, S offers } P = \frac{1}{1+\delta}, \text{ B accepts if } P \leq \frac{1}{1+\delta}.$$

$$\text{In every even period, B offers } P = \frac{\delta}{1+\delta}, \text{ S accepts if } P \geq \frac{\delta}{1+\delta}.$$

- Though this is not a rigerous way to determine the equilibrium and the proof to show it is unqiue is tedious (See Ausubel Notes), we can show that it is a SPE. First note that the game starting periods 1 and 3 are identical as well as the games starting in period 2 and 4.

- So consider a game starting in an odd period where S offers $P = \frac{1}{1+\delta}$:

  - Does B have a profitable deviation? If B accepts: payoff $= 1 - \frac{1}{1+\delta} = \frac{\delta}{1+\delta}$. If B rejects: payoff $= \delta * \frac{1}{1+\delta} = \frac{\delta}{1+\delta}$.
  - Does S have a profitable deviation? If S offers $P = \frac{1}{1+\delta}$, payoff $= \frac{1}{1+\delta}$. If S offers $P > \frac{1}{1+\delta}$, B will reject and offer a Nash bid yielding a payoff to S of $\delta\frac{\delta}{1+\delta} = \frac{\delta^2}{1+\delta} < \frac{1}{1+\delta}$. So S would be strictly worse off.

  Thus B and S do not have profitable deviations in odd periods.

- So consider a game starting in an even period where B offers $P = \frac{\delta}{1+\delta}$:

  - Does S have a profitable deviation? If S accepts: payoff $= \frac{\delta}{1+\delta}$. If S rejects: payoff $= \delta * \frac{1}{1+\delta} = \frac{\delta}{1+\delta}$.

– Does B have a profitable deviation? If B offers $P = \frac{\delta}{1+\delta}$, payoff $= 1 - \frac{\delta}{1+\delta} = \frac{1}{1+\delta}$. If B offers $P < \frac{\delta}{1+\delta}$, S will reject and offer a Nash bid yielding a payoff to B of $\delta(1 - \frac{1}{1+\delta}) = \frac{\delta^2}{1+\delta} < \frac{1}{1+\delta}$. So B would be strictly worse off.

Thus B and S do not have profitable deviations in even periods.

- Thus we have shown that the strategies above constitute a NE in every subgame and therefore constitute a SPE.

23 Lecture 23: November 16, 2004

23.1 Cooperative Games

- First note that an infinite horizon game is not repeated at all, it simply has a possibly very long horizon but it could also end tomorrow. It may never end but you might think of it as an auction for ONE item which never reaches an agreement.
- An infinitely repeated game involves a different item each time and the same game is repeated.
- So far we have considered non-cooperative games in which players act only for themselves and the game is well laid out. We now turn to cooperative games where the structure is not as formal and we don't know exactly what form of cooperation will occur, but we can make some reasonable assumptions about what the solution will look like.

Nash Bargaining Solution (NBS)

- Consider the payoff graph in G-23.1. Here we represent player 1's and player 2's payoffs and draw in a "Feasible Set" of payoff combinations. The solution to the problem will be within this area. We also designate a Disagreement Point which results if no agreement is decided upon. From that we can limit the feasible set to those payoffs which are greater than or equal to the disagreement point as shown in the graph.
- See G-23.2 for a graph of the alternating offer game we discussed previously. Here, the disagreement point is at $d = (0, 0)$ but there is a certain symmetry in the game.
- Axioms of a "Reasonable" Solution:
  - (1) The solution should not depend on linear transformations of player's utility functions.
  - (2) The solution should be individually rational and pareto-optimal.
  - (3) There should be "Independence of Irrelevant Alternatives."
  - (4) The solution should be symmetric if the game itself is symmetric.
- So note that axioms (2) and (4) give us the solution E_1 in graph G-23.2. The solution set, with a possible point, E_2, in G-23.1 is a possible Nash Bargaining Solution.
- A note on axiom (3). This means that say we set up the problem and solve for the NBS. Then if we were to remove part of the feasible set (that does not include either the solution or the disagreement point), then we should also get the same NBS in the new problem.

- **Theorem:** Suppose a feasible set is convex, closed and bounded above, then there exists a unique solution satisfying the four axioms and it is given by:

$$Max_{x \geq d}(x_1 - d_1)(x_2 - d_2).$$

Where $d = (d_1, d_2)$ is the disagreement point and $x = (x_1, x_2) \in$ feasible set.

- Note we don't necessarily make assumptions about bargaining power or if the NBS is really going to happen because we don't specify the setup in this game. All we have done is characterize a "reasonable" equilibrium.

23.2 Repeated Games

- **Definition:** Let G be a static game. Then the T-period repeated game denoted $G(T, \delta)$ consists of the game G repeated T times. At each period t, the moves of all players in all previous periods are known to all players. Payoff to player i is then:

$$u_i = \sum_{t=1}^{T} \delta^{t-1} u_{it}.$$

If $T = \infty$, $G(T, \delta)$ is an infinitely repeated game and the AVERAGE payoff to player i is:

$$u_i = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u_{it}, \quad \delta < 1.$$

Where $(1 - \delta)$ is just a normalization term which allows us to compare the payoff in the infinite game to the finite or static game.

Trigger Strategy Equilibria

- First define a "main equilibrium path" to be an action suggested for every player i, $(i = 1, \ldots, n)$, and for every period:

$$\vec{s} = \underbrace{(s_{11}, \ldots, s_{n1})}_{Period\ 1}, \underbrace{(s_{12}, \ldots, s_{n2})}_{Period\ 2}, \underbrace{(s_{13}, \ldots, s_{n3})}_{Period\ 3}, \ldots$$

Second, define $(s_1^*, \ldots, s_n^*)$ to be the NE of the static game, G. Then a Trigger Strategy for player i in the repeated game $G(T, \delta)$ is given by:

$$\sigma_{it} = \begin{cases} s_{it}, & \text{if every player has played according to } \vec{s} \text{ in all previous periods (or } t = 1) \\ s_i^*, & \text{if a deviation has occurred in any previous period by any player} \end{cases}$$

- For δ high enough, we will show that these trigger strategies constitute a SPE.

24 Lecture 24: November 18, 2004

24.1 Trigger Equilibria

- For a trigger strategy equilibrium to be subgame perfect, we must show that both on the main equilibrium path and the punishment path, there is no incentive to deviate at any point in time.
- Clearly, if the punishment strategy is the static NE, then we're ok on part of that.
- So show the other condition, we need to do some work. Note the trigger occurs even if YOU are the one to deviate.
- Consider the example of the infinitely repeated cournot game with 2 players. Consider the trigger strategies for each player:

$$q_{it} = \begin{cases} \frac{a-c}{4} & if\ q_{1s} = q_{2s} = \frac{a-c}{4}\ \forall\ s = 1, \ldots, t-1 \\ \frac{a-c}{3} & else. \end{cases}$$

So we play the joint monopoly solution, splitting $q^m = \frac{a-c}{2}$ along the main equilibrium path, and then play the static Nash of $q = \frac{a-c}{3}$ as a punishment.

- Payoff along the equilibrium path (with no deviations):

$$\pi^E = \sum_{t=0}^{\infty} \delta^t \frac{a-c}{4}(a - \frac{a-c}{4} - \frac{a-c}{4} - c) = \frac{1}{1-\delta}\frac{1}{8}(a-c)^2.$$

- Payoff from deviating in period 1:

$$\pi^D = \underbrace{\frac{3(a-c)}{8}(a - \frac{a-c}{4} - \frac{3(a-c)}{8} - c)}_{Deviation\ Payoff} + \underbrace{\sum_{t=1}^{\infty} \delta^t \frac{a-c}{3}(a - \frac{a-c}{3} - \frac{a-c}{3} - c)}_{Punishment\ Payoff} =$$

$$= \frac{9}{64}(a-c)^2 + \frac{\delta}{1-\delta}\frac{1}{9}(a-c)^2.$$

- Note we found the deviation payoff by $Max_{q_i|q_j=(a-c)/4}\pi_i = q_i(a - (a-c)/4 - q_i - c)$.
- When will the equilibrium path be attained? (Note that the game looks the same in every period with the same strategies so considering a deviation at period 2 is exactly the same as considering a deviation at period 1). We need $\pi^E \geq \pi^D$. Or:

$$\frac{1}{1-\delta}\frac{1}{8}(a-c)^2 \geq \frac{9}{64}(a-c)^2 + \frac{\delta}{1-\delta}\frac{1}{9}(a-c)^2.$$

$$\frac{1}{1-\delta}\frac{1}{8} \geq \frac{9}{64} + \frac{\delta}{1-\delta}\frac{1}{9}.$$

$$\hat{\delta} \geq \frac{9}{17}.$$

So the players must not discount the future too much. Note a $\delta = 0$ would imply the future was worthless so you would alway deviate immediately. A $\delta = 1$ means you would definitely stay along the main path, but in general,

$$\hat{\delta} \in (0, 1).$$

Players must be sufficiently patient.

- **Definition:** An n-tuple of payoffs $(x_1, \ldots, x_n)$ to the n players is called feasible if it arises from the play of pure strategies or if it is a convex combination of payoffs of pure strategies. In a 2 player $2x2$ game (like prisoners dilema), plot the payoff combinations in payoff space and connect and fill in. This is the feasible set.

- **Theorem:** Folk Theorem. Let $(e_1, \ldots, e_n)$ be the payoffs from a NE of the game G and let $(x_1, \ldots, x_n)$ be any feasible payoffs from G. If $x_i > e_i \; \forall \; i$ then $\exists$ a SPE of $G(\infty, \delta)$ that attains $(x_1, \ldots, x_n)$ as the AVERAGE payoffs provided that δ is close enough to 1.

- See G-24.1 and G-24.2 for cournot and prisoners dilema. We draw the feasible set of payoffs, include the (e_1, e_2) point and construct the Folk Theorem Region as shown. Notice we do not say for sure what the payoff is going to be under Folk, but we simply know that we can construct a set of trigger strategies with a large enough delta to sustain any payoff outcome in the Folk Theorem Region.

25 Lecture 25: November 23, 2004

25.1 Repeated Prisoner's Dilemma

- Consider the payoff space for the PD game in G-25.1. Note the folk region contains the point $(-1,-1)$. We can obtain that (average) payoff in $G(\infty,\delta)$ if:

$$\sum_{t=0}^{\infty}\delta^t(-1) \geq 0 + \sum_{t=1}^{\infty}\delta^t(-4).$$

$$\frac{-1}{1-\delta} \geq \frac{-4\delta}{1-\delta}.$$

$$\delta \geq \frac{1}{4}.$$

- We could also support $(-0.5,-3)$ as a SPE by choosing appropriate strategies.
  - Step 1: Write the payoff as a convex combination of payoffs from pure strategies:

$$(-0.5,-3) = 0.5*(-1,-1) + 0.5*(0,-5).$$

  - Step 2: Define the main equilibrium path that would yield the required payoff. One might be:

$$(RS, RS) - (C, RS) - (RS, RS) - (C, RS) - (RS, RS) - (C, RS) - \cdots.$$

    We alternate between the strategies that give us payoffs on either end of the convex combination.
  - Step 3: Write out strategies:

$$\sigma_{1t} = \begin{cases} RS & \text{if } t \text{ is odd and no prior deviations} \\ C & \text{if } t \text{ is even or there was a prior deviation} \end{cases}$$

$$\sigma_{2t} = \begin{cases} RS & \text{if no prior deviations} \\ C & \text{otherwise} \end{cases}$$

  - Step 4: Find the sustainable δ. We would have to check 4 conditions: Player (I,II) x Period (Odd,Even).

- Note we could also use a mixed strategy but the randomization mechanism must be public. If it is private, the player may have an incentive to cheat and be completely undetected.

25.2 Maximally Collusive Equilibria

- Can we develop more sophisticated strategies to yield a lower required discount factor? Yes, consider the repeated Cournot game and the flow diagram in G-25.2.

- We start with collusion and each firm produces one half the monopoly quantity $(a-c)/4$. If there is a deviation, we don't revert to cournot nash, $(a-c)/3$, but instead to $q_{punish} = (a-c)/2$. This is WAY too much output and results in negative profits for both firms. We continue like this until the original deviator also sets $q = (a-c)/2$ for one period (which is like a signal to the other firm that it wants to collude again) and in the next period, we have collusion.

- Hence the punishment is much more HARSH than in the trigger strategy, however the length of the punishment is very SHORT compared to the trigger.

- Hence the critical discount factor in this setup is lower than in the trigger setup. Why? Because in the trigger we revert to cournot nash forever so some of the punishment is discounted into the future. With this new setup, we punish IMMEDIATELY! so even firms with a relatively low discount factor (they do not value the future as much) would be willing to collude in every period.

26 Lecture 26: November 30, 2004

26.1 Static Games of Incomplete Information

- We will focus on the study of auctions for this part of the material. First some definitions.
- **Definition:** First Price Auction. Every player i simultaneously submits a bid of b_i. Player i wins the item if he has the highest bid and then pays a price of b_i.
- **Definition:** Second Price Auction. Every player i simultaneously submits a bid of b_i. Player i wins the item if he has the highest bid and then pays a price of b_j where b_j is the second highest bid.
- **Definition:** English Auction. Ascending dynamic bids where winner pays amount of highest bid. (Christies and E-Bay).
- **Definition:** Dutch Auction. Descending Price Auction. Start high and lower the bid. First bidder to claim the item wins and pays that price.
- Example: Auction with Discrete Bids. Suppose there is one item up for auction in a sealed bid, first price auction. There are only two allowable bids: 0 and 1/3. There are two risk-neutral bidders each with private valuation,

  $$t_i \sim U[0, 1].$$

  Of course each bidder knows his own valuation but not the other. He only knows the distribution of the other's valuations. The highest bidder wins and pays his bid. In the case of a tie, a coin is flipped to determine the winner.
- A strategy in this case is a mapping from your "type" to an "action." In this case, we map from valuations to bids. Players bid against distributions of bids since true opponent's valuations are unknown.
- Solution. Every Bayesian Nash Equilibrium has the following form:

  $$S_1^*(t_1) = \begin{cases} 0 & \text{if } t_1 \in [0, \hat{t}_1] \\ 1/3 & \text{if } t_1 \in (\hat{t}_1, 1] \end{cases}$$

  $$S_2^*(t_2) = \begin{cases} 0 & \text{if } t_2 \in [0, \hat{t}_2] \\ 1/3 & \text{if } t_2 \in (\hat{t}_2, 1] \end{cases}$$

  So a strategy for player i is contingent on what his valuation is realized to be. See G-26.1. For each player, we define a break-point, $\hat{t}_i$ such that if his realized valuation is above this break-point, he bids 1/3, else he bids 0. Note that $\hat{t}_1$ need not equal $\hat{t}_2$ but of course in this symmetric game, they will be equal. So far, all we can say is that $\hat{t}_i > 1/3$. Why? Because if it was below and his realized valuation was between the break-point and 1/3, he might be bidding 1/3 for an item which he valued less than 1/3.

- What do we know about the solution? The BNE must satisfy:

$$S_1^* = arg\ max\ E[u_1(S_1, S_2^*(t_2); t_1)]\ \forall\, t_1 \in [0,1],$$

and,

$$S_2^* = arg\ max\ E[u_2(S_1^*(t_1), S_2; t_2)]\ \forall\, t_2 \in [0,1].$$

- So what is player 1's (for example) expected payoff from bidding 1/3?

$$\begin{aligned} E[u_1(1/3, S_2^*(t_2); t_1)] &= \underbrace{Pr(\{t_2 > \hat{t}_2\})}_{P2\ Bids\ 1/3} * \underbrace{0.5(t_1 - 1/3)}_{Coin\ Flip\ Payoff} + \underbrace{Pr(\{t_2 \le \hat{t}_2\}) * (t_1 - 1/3)}_{P1\ wins}. \\ &= (1 - \hat{t}_2) * 0.5(t_1 - 1/3) + \hat{t}_2 * (t_1 - 1/3) \\ &= [0.5 * (1 - \hat{t}_2) + \hat{t}_2] * (t_1 - 1/3) \end{aligned}$$

- So what is player 1's (for example) expected payoff from bidding 0?

$$\begin{aligned} E[u_1(0, S_2^*(t_2); t_1)] &= \underbrace{Pr(\{t_2 > \hat{t}_2\})}_{P2\ Bids\ 1/3} *0 + \underbrace{Pr(\{t_2 \le \hat{t}_2\}) * 0.5(t_1 - 0)}_{P1\ wins}. \\ &= 0.5 * \hat{t}_2 t_1 \end{aligned}$$

- Observe that for $t_1 > \hat{t}_1$, player 1's expected utility from playing 1/3 is higher than that for bidding 0. The opposite is also true. By continuity, it must be the case that:

$$E[u_1(1/3, S_2^*(t_2); \hat{t}_1)] = E[u_1(0, S_2^*(t_2); \hat{t}_1)].$$

Plugging in $t_1 = \hat{t}_1$ and setting equal, we have:

$$\begin{aligned} (0.5(1 - \hat{t}_2) + \hat{t}_2)(t_1 - 1/3) &= 0.5\hat{t}_2 t_1 \\ (0.5(1 - \hat{t}_2) + \hat{t}_2)(\hat{t}_1 - 1/3) &= 0.5\hat{t}_2 \hat{t}_1 \\ \hat{t}_1 &= 1/3\hat{t}_2 + 1/3 \end{aligned}$$

- This argument could be repeated for player 2, but since the game is symmetric:

$$\hat{t}_2 = 1/3\hat{t}_1 + 1/3.$$

Two equations, two unknowns yields:

$$\hat{t}_1 = \hat{t}_2 = \frac{1}{2}.$$

- This defines the bidding strategies. Note that both players will also "know" this information going into the game. The only source of incomplete information is the other player's actual realization of his valuation.

- It might seem more intuitive that $\hat{t}_i = 1/3$. However, it is not just each player's own valuation that matters but the incomplete information regarding his opponent's valuation. Note a player does not bid 1/3 when his valuation is between 1/3 and 1/2. Why? Because his gains from winning the auction are too small compared with bidding 0 and possibly still having a tie and winning the item on a coin flip. The same argument would apply if the discrete bid choices where $(0, 2/3)$. In this case, the players would always bid ZERO! Why? Because from our previous argument, their break-point would clearly be at least 2/3. Suppose it is exactly 2/3. Then if they realized a valuation of 3/4, they would win $(3/4 - 2/3)$ if they won the auction. This is too small compared with the expected value of bidding 0. In fact, any discrete bidding requirement larger than $(0, 1/2)$ would result in no bidding. Why is this Nash? If this information is known to both players, then clearly one could unilaterally deviate and win the auction with certainty. But would the bidder really be better off? Suppose discrete bids were $(0, 2/3)$ and the player realized a valuation of 1. He would win (with certainty) 1/3 from bidding 2/3 (since the other player is bidding 0). However, by bidding 0, he gets (via the coin flip):

$$0.5 * (1 - 0) = 0.5.$$

  This is clearly higher than bidding 2/3, winning, and getting a payoff of 1/3. So all this hinges importantly on the fact that these players are risk-neutral. If the players were a bit more risk averse, one might prefer the sure 1/3 payoff to the 50/50 chance of getting 1 or 0.

27 Lecture 27: December 2, 2004

27.1 Bayesian Nash Equilibrium

- **Definition:** Let T_1, T_2 be the sets of possible types for players 1 and 2. Define (S_1^*, S_2^*) to be the Bayesian Nash Equilibrium (BNE) if $\forall\ t \in T$,

$$S_1^*(t_1) = arg\ max\ _{a_1 \in A_1} \underbrace{\sum_{t_2 \in T_2} u_1(a_1, S_2^*(t_2); t_1) \cdot p_1(t_2|t_1)}_{E[u_1]}.$$

$$S_2^*(t_2) = arg\ max\ _{a_2 \in A_2} \underbrace{\sum_{t_1 \in T_1} u_2(S_1^*(t_1), a_2; t_2) \cdot p_2(t_1|t_2)}_{E[u_2]}.$$

  Where S_i^* is a function from types to actions for player i, or usually valuations to bids in an auction setting. Note we have assumed only two players and finitely many types, but both of these assumptions can be relaxed.

- Note that we have the conditional probability in the definition of a BNE because usually types are correlated. (Ie, if I think the piece of art up for auction is a piece of crap, odds are, so do you).

Solution to the Sealed-Bid First-Price Auction

- Consider a two player game where $v_i \sim U[0,1]$ and $b_i \in [0,1]$. Bidders simultaneously and independently choose b_i once they have realized their own v_i. The other's valuation is unknown. The highest bidder wins the item and pays her bid.

- Assume the bidding function is increasing in v_i and the bidding function is the same for each player.

- Bidder i wins if:

$$b_i > B(v_j) \Longrightarrow B^{-1}(b_i) > v_j \Longrightarrow Pr(B^{-1}(b_i) > v_j) = B^{-1}(b_i),$$

  since we assumed the valuations where uniform $[0,1]$.

- Define the expected payoff to bidder i when her valuation is v_i as:

$$\pi_i(v_i, b_i) = (v_i - b_i)B^{-1}(b_i).$$

  And also define the expected payoff to bidder i from bidding optimally:

$$\Pi_i(v_i) = Max_{b_i}\ \pi_i(v_i, b_i).$$

Note the arg max of this last expression is $B(v_i)$, the optimal bid. Thus,

$$\Pi_i(v_i) = \pi_i(v_i, B(v_i)) = (v_i - B(v_i))B^{-1}(B(v_i)) = (v_i - B(v_i))v_i. \quad (1)$$

- We can also calculate $\Pi_i(v_i)$ another way. Note:

$$\frac{d\Pi_i}{dv_i} = \left.\frac{\partial \pi_i(v_i, b_i)}{\partial v_i}\right|_{b_i=B(v_i)} = B^{-1}(v_i)\Big|_{b_i=B(v_i)} = v_i.$$

Here we used the envelope theorem. Now write the difference between Π at two valuations as the integral of its derivative:

$$\Pi_i(v_i) - \Pi_i(0) = \int_0^{v_i} \frac{d\Pi_i}{dv_i} dv_i = \int_0^{v_i} v_i dv_i = \left.\frac{1}{2}v_i^2\right|_0^{v_i} = \frac{1}{2}v_i^2. \quad (2)$$

- Note that $\Pi_i(0) = 0$ so $\Pi_i(v_i) = 1/2v_i^2$. Setting (2) equal to (1),

$$(v_i - B(v_i))v_i = \frac{1}{2}v_i^2.$$

$$v_i - B(v_i) = \frac{1}{2}v_i.$$

$$B(v_i) = \frac{1}{2}v_i.$$

And this is our BNE. A bidding strategy from the player's valuations to her bids. See G-27.1 and G-27.2. Note that she always shades her bid to one half her actual valuation in a first priced auction with two players.

- If there were N players, the BNE would be:

$$B(v_i) = \frac{N-1}{N}v_i.$$

Note as $N \to \infty, B(v_i) \to v_i$.

28 Lecture 28: December 7, 2004

28.1 Second Price Sealed Bid Auction

- Winners pays second highest bid. We can determine the NE strategy by considering the following table:

		Bidder i		
	Payoff If	Shades ($b_i' < v_i$)	Sincerely ($b_i = v_i$)	Inflates ($b_i'' > v_i$)
1	$\hat{b}_{-i} \leq b_i'$	$v_i - \hat{b}_{-i}$	$v_i - \hat{b}_{-i}$	$v_i - \hat{b}_{-i}$
2	$b_i' < \hat{b}_{-i} < v_i$	0	$v_i - \hat{b}_{-i} > 0$	$v_i - \hat{b}_{-i}$
3	$v_i < \hat{b}_{-i} \leq b_i''$	0	0	$v_i - \hat{b}_{-i} < 0$
4	$\hat{b}_{-i} > b_i''$	0	0	0

- So it is clear that sincere bidding weakly dominates shading and inflating.

28.2 Dynamic Auctions

- English versus Second Price Sealed Bid. We have shown that in a second price sealed bid auction, bidders bid their true valuations. In the end, the bidder with the highest valuation wins but ends up paying only the valuation of the second highest bidder. In an ascending clock English auction, the clock stops when there is only one bidder still in the auction. Again, this will occur as soon as the going bid goes above the second highest bidder. Hence the payoff is the same. With private independent valuations, the revenue generated from a English and Second Price Seal Bid auction is the same (Revenue Equivalence).

- Dutch versus First Price Sealed Bid. Recall that in a Dutch auction, the price starts high and falls and the first bidder to ring in, wins the item and pays the price he bid. In this setting, it is impossible for bidders to gain any information about their opponents valuations. When someone bids, you get a bit of information, but it's too late, the auction is over. Strategically, this is equivalent to the first price auction and it can be shown that the nash strategy is to bid:

$$b_i = \frac{n-1}{n} v_i.$$

- The English and Second Price Sealed Bid, though they yield the same revenue, are not really identical, since in an English auction, you are gaining information about the others valuations as the auction goes on. In theory, you could continually revise your valuation based on the number of bidders still active in the auction. The presence of independent private valuations is really what is driving this revenue equivalence. The auctions are really very different otherwise.

- Summary Items on Auctions.
  - (1) First price sealed bid auction: shade your bid to $(n-1)/n * v_i$.
  - (2) Second price sealed bid auction: bid your valuation.
  - (3) English auction: bid up to your valuation.
  - (4) In common value auctions, the item up for auction has a common value V, but each bidder only has a noisy estimate of V, say $V_i + \epsilon_i$, where $\epsilon_i \sim (0, \sigma^2)$. The winner is the bidder with the highest error term!
  - (5) The so called "Winner's Curse" refers to the idea that winning the item up for auction (in common value auctions) conveys some bad news to the winner that everyone else thought that item was worth less than you! The savy bidder will take this into account and shade down his valuation. Note this is included in bidder's strategies so the equilibrium payoff is still positive! Otherwise, no one would participate in the auction!
  - (6) Bidders in an English auction have a smaller "curse" than a second price sealed bid auction because of the information bidders receive from the other bidder's behavior.
  - (7) In a common value auction, an ascending bid auction yields a higher expected payoff to the seller than a sealed bid auction because the winner's curse effect is lowered.

28.3 Auctions for Many Items

- Usually, as in treasury auction, bidders submit entire demand schedules for items instead of a round by round bidding process to save time.
- Consider the Ausubel Auction for 5 identical items where 4 bidders marginal values for having one, two, and three of the items are as follows:

Player I	Player II	Player III	Player IV
$V_1 = 123$	$V_1 = 125$	$V_1 = 75$	$V_1 = 85$
$V_2 = 113$	$V_2 = 125$	$V_2 = 5$	$V_2 = 65$
$V_3 = 103$	$V_3 = 49$	$V_3 = 3$	$V_3 = 7$

- We start the price low and raise it and the 4 players continually revise the number of items they would like. Stop when supply equals demand.
- So at a price of 50, I wants 3, II wants 2, III wants 1, and IV wants 2.
- At a price of 75, I wants 3, II wants 2, III wants 0, and IV wants 1.

- At this point demand is 6 and supply is 5. Now we ask, should the price continue to rise above 85 so only players I and II receive the items? Consider player I's payoff from letting the price rise and attaining 3 items:

$$\pi_1 = (123 - 85) + (113 - 85) + (103 - 85) = 84.$$

  And from dropping his demand from 3 to 2 when the price hits 75:

$$\pi_1 = (123 - 75) + (113 - 75) = 86.$$

  So player I should drop out at a price of 75, thereby giving player IV an item but maximizing his payoff. Revenue is $75 * 4 = 300$.

- This is a problem since the items should have all gone to players I and II but instead we have an inefficient allocation.

- Consider Ausubel's alteration: We award a good to a player as soon as they have clinched at whatever price the bidding is currently at.

- At a price of 64, player I's opponents demand 5 units, so it is still possible that player I won't get any units. At a price of 65, player IV drops to 1 unit of demand so the total demand of player I's opponents is 4 units. Hence player I is guaranteed one unit at price of 65. He has clenched a unit at P = 65. The same happens at 75 and 85 and in this case, it is in player I's interest to win all three units because of the non-uniform pricing. Player I's payoff becomes:

$$\pi_1 = (123 - 65) + (113 - 75) + (103 - 85) = 144.$$

  And total revenue is $65 + 75 + 75 + 85 = 300$. Revenue is the same but the allocation is now more efficient. (Note player II clinched a unit at 75).

29 Lecture 29: December 9, 2004

29.1 Sealed Bid Double Auction

- See G-29.1. We have two bidders with valuations $V_b, V_s \sim U[0,1]$ where both the buyer and seller have incomplete information about the other's valuation. It can be shown (see problem set) that trade occurs if:

$$V_b \geq V_s + 1/4.$$

  This is inefficient because we should have trade as long as the buyer's valuation is higher than the sellers. The lack of information leads to this inefficiency.

29.2 Dynamic Games of Incomplete Information

- **Definition:** Signaling: Engaging in a costly activity for the purpose of credibly convincing your opponent of your type.
- Denote the "Sender", the one who observes the private information and then moves (signals). While the "Reciever" lacks the information, but may be able to infer it from the sender's move.
- Example 1) The management at a firm might let a strike occur to signal to the labor union that they are indeed in a period of low profits. The union may not believe the claim, but the willingness to let the strike happen is a costly signal by the firm and it would be TOO costly for a firm which actually was making high profits.
- Example 2) A firm might issue dividends to signal to shareholders that profits were high. Issuing dividends versus buying back their own shares (and pushing up their stock price) is more costly and is a practice that a low profit firm would not be willing to engage in because they simply did not have the money!
- Example 3) A firm might pay millions for an endorsement of a product which really does not convey anything about the quality of the product. The signal is that the firm is saying our product is that good that if this endorsement entices you to try the product just once, you'll keep buying it.
- An important part of dynamic games is the order of events:
  - (1) First, nature selects the type, $t \in T$, for the sender according to a privately known probability distribution.
  - (2) Then the sender observes t and choose a message, $m \in M$, to send to the receiver.
  - (3) The receiver receives m and then selects an action, $a \in A$, that affects both the sender and receiver. Note that the sender will send a different message depending on t, and the receiver may update his probability of t once he receives m.
  - (4) Payoffs are realized: $U_s(t, m, a)$ and $U_r(t, m, a)$.

Job Market Signaling

- Now the sender is a worker and the receivers are firms. Productivity, η, of the worker is either high (H) or low (L) with probability $(q, 1-q)$. The message is a choice of education level, $e \geq 0$.
- Suppose 2 firms observe e and update their beliefs on η. Then firms offer wages. The worker will accept the higher wage and payoffs are:

$$U_w = w - c(\eta, e), \quad U_f = y(\eta, e) - w.$$

  Where f is the winning firm and y is the marginal product of a worker of type η and with education e.

- Note that education could be completely worthless for increasing productivity, but the investment is still a signal of the worker's type.
- See G-29.2 for a non-cynical (education matters) and a cynical graph (education is worthless) of productivity as a function of type and education.
- The requirements for a "Sequential Equilibrium" or a "Perfect Bayesian Equilibrium" are the following:
  - (1) Beliefs: R maintains a probability distribution over types - so R updates beliefs after each move.
  - (2) Updating by Bayes Rule.
  - (3) Sequential Rationality: Each player must be optimizing according to his beliefs and the information he has. Thus R's choice of action must maximize his expected utility and S's choice of message must maximize his utility given his knowledge of t as well as his anticipation of R's response to m.
- There are 3 types of equilibria: Pooling, Separating and Hybrid.

Pooling Equilibria

- Both types of workers choose a common education level, e_p. The firm's beliefs after observing e_p are the same as his prior beliefs:

$$\eta = \begin{cases} H, & \text{with probability } q \\ L, & \text{with probability } 1-q \end{cases}$$

Separating Equilibria

- The H type worker chooses $e = e_s$ and the L type worker chooses $e = e_L$. The firm's beliefs after observing e_s are :

$$\eta = \begin{cases} H, & \text{with probability } 1 \\ L, & \text{with probability } 0 \end{cases}$$

And after observing e_L,

$$\eta = \begin{cases} H, & \text{with probability } 0 \\ L, & \text{with probability } 1 \end{cases}$$

Hybrid or Partially Pooling Equilibria

- The H type worker chooses $e = e_h$ always and the L type worker chooses $e = e_h$ with probability π and $e = e_l$ with probability $1 - \pi$. The firm's beliefs after observing e_h must be updated using Bayes rule:

$$Pr(H|e_h) = \frac{Pr(e_h|H) \cdot Pr(H)}{Pr(e_h|H) \cdot Pr(H) + Pr(e_h|L) \cdot Pr(L)} = \frac{q}{q + \pi(1-q)} \equiv q'.$$

$$Pr(L|e_h) = \frac{Pr(e_h|L) \cdot Pr(L)}{Pr(e_h|L) \cdot Pr(L) + Pr(e_h|H) \cdot Pr(H)} = \frac{\pi(1-q)}{q + \pi(1-q)} \equiv 1 - q'.$$

So the firm's updated beliefs after observing e_h are:

$$\eta = \begin{cases} H, & \text{with probability } q' \\ L, & \text{with probability } 1 - q' \end{cases}$$

And after observing e_L,

$$\eta = \begin{cases} H, & \text{with probability } 0 \\ L, & \text{with probability } 1 \end{cases}$$

Solution to the Pooling Equilibrium

- If e_p is observed, since the firm cannot update their probability about the worker's type, they offer:

$$w(e_p) = qy(H, e_p) + (1-q)y(L, e_p),$$

where q and $1 - q$ are the prior probabilities.

Solution to the Separating Equilibrium

- First note that though we only choose two levels of education, it is possible that the firm will observe an education signal different from e_L or e_s. Thus you could assume a high type for any education of e_s or greater and a low type for a signal below e_s.

- The firm will offer wage:

$$w(e) = \begin{cases} y(H, e), & e \geq e_s \\ y(L, e) & e < e_s \end{cases}$$

- We need some self selection constraints (or compatibility constraints) to make sure that each type of worker only chooses each level of education. Thus,

$$Low: \;\; w(e_L) - c(L, e_L) \geq w(e_s) - c(L, e_s).$$

$$High: \;\; w(e_s) - c(H, e_s) \geq w(e_L) - c(H, e_L).$$

- We also know that under complete information, a low and high type would maximize such that:

$$e_L^* = arg\ max \;\; Max_{e_L}\{y(L, e) - c(L, e)\},$$

  and,

$$e_H^* = arg\ max \;\; Max_{e_H}\{y(H, e) - c(H, e)\}.$$

  We claim that $e_L = e_L^*$ and $e_s > e_H^*$. It is clear that the low type should definitely not get more education than exactly what is necessary to be called a low type (possibly 0) but they also shouldn't get any less because e_L^* is optimal. Under the cynical graph, the unconstrained choice for a high type would be an education level of 0, but this would mean that a low type could easily jump up and "look like" a high type. This is also true if $e_s = \epsilon$. The low type could incur a relatively small cost and "look like" a high type. Thus e_s must be large enough to make the low type just indifferent between getting education e_L and education e_s.

- Thus, there is a systematic bias where high type workers are getting TOO much education when it is not increasing productivity, but rather just creating a signal to the firms.

Solution to the Hybrid Equilibrium

- The firm can now update their probability distribution of the worker's type based on their education signal. Thus, they offer:

$$w(e) = \begin{cases} q'y(H, e_h) + (1 - q')y(L, e_h), & e \geq e_h \\ y(L, e_L) & e < e_h \end{cases}$$

- Self selection constraints:

$$Low: \;\; w(e_L) - c(L, e_L) = w(e_h) - c(L, e_h).$$

$$High: \;\; w(e_h) - c(H, e_h) \geq w(e_L) - c(H, e_L).$$

  Now the low type is indifferent between the different education signals.
  Done.

Review for Final

29.3 Price Discrimination

- 3^{rd} degree: non-uniform/linear pricing. Need the ability to identify markets, no resale, and a mechanism for voluntary self-selection. Set prices as if you are operating in two separate markets.
- 2^{nd} degree: uniform/non-linear pricing. 2 part tariff. Menu is offered such that utility satisfies individual rationality and incentive compatibility constraints. High value consumer will buy at efficiency quantity ($u'(x) = c$) while the low value pays a higher price for less quantity.
- 1^{st} degree: non-uniform/non-linear pricing. Menu offered (2 part tariff). We can identify which consumer is high valued, so no need for IC constraint. Just rationality.

29.4 Game Theory

- A NE is a set of mutually best response strategies.
- Cournot quantity with 2 playes and $P = a - Q : q_i = (a - c)/3$. Monopoly is $(a - c)/2$.
- When checking for NE, only look at unilateral deviations.
- IF IESDS yields a unique strategy n-tuple, this is the unqiue strict NE. Every NE survives IESDS. IESDS is order-independent.
- If a player is mixing between two strategies, they must be indifferent between the two (they must yield the same expected payoff).
- Nash Existance Theorem: Every finite game has at least one NE (possibly in mixed strategies).
- Brouwer: X is non-empty, compact and convex. $f : X \mapsto X$ is continuous. $\Rightarrow f$ has a fixed point.
- Kakutani: X as in Brouwer, $F : X \mapsto X$ non-empty, convex valued correspondence with a closed graph. $\exists\ x \ni x \in F(x)$.
- With X as a set of strategies and F as the best response correspondence, via Kakutani, there exists a vector of mixed strategies with the property that all players are playing a best response - the fixed point guaranteed by Kakutani is a NE.
- Subgame Perfect NE - a set of strategies that constinute a NE in every subgame.
- Stackelberg - lead produces $(a - c)/2$ (monopoly quantity) while follower produces $(a - c)/4$.

- Nash Bargaining Solution: Axioms of a reasonable solution: (1) solution should not depend on linear transformations for player's utility functions. (2) Individually rational and pareto-optimal. (3) Independence of Irrelevant Alternatives. (4) Symmetric is G is symmetric.

- NBS: $Max_{x \geq d}\ (x_1 - d_1)(x_2 - d_2)$.

- Feasible Payoff: a payoff that arises from play of pure strategies or convex combinations of those strategies.

- Folk Theorem. Let (e_1, e_2) be the NE payoffs and let (x_1, x_2) be any feasible set of payoffs with $x_i > e_i$. Then $\exists$ a SPE of $G(\infty, \delta)$ that attains (x_1, x_2) as the average payoff provided that δ is sufficiently close to 1.

- Bayesian Nash Equilibrium

$$S_1^*(t_1) = arg\ max\ _{a_1 \in A_1} \underbrace{\sum_{t_2 \in T_2} u_1(a_1, S_2^*(t_2); t_1) \cdot p_1(t_2|t_1)}_{E[u_1]}.$$

- Sealed-Bid First-Price Auction. Define:

$$\pi_i(v_i, b_i) = (v_i - b_i)B^{-1}(b_i).$$

$$\Pi_i(t_i) = Max_{b_i}\ \pi_i(t_i, b_i).$$

Thus,

$$\Pi_i(t_i) = \pi_i(v_i, B(v_i)) = (v_i - B(v_i))B^{-1}(B(v_i)) = (v_i - B(v_i))v_i. \quad (1)$$

$$\frac{d\Pi_i}{dv_i} = \left.\frac{\partial \pi_i(v_i, b_i)}{\partial v_i}\right|_{b_i = B(v_i)} = \left. B^{-1}(v_i)\right|_{b_i = B(v_i)} = v_i.$$

$$\Pi_i(v_i) - \Pi_i(0) = \int_0^{v_i} \frac{d\Pi_i}{dv_i} dv_i = \int_0^{v_i} v_i dv_i = \left. \frac{1}{2} v_i^2 \right|_0^{v_i} = \frac{1}{2} v_i^2. \quad (2)$$

$$B(v_i) = \frac{1}{2} v_i.$$

And this is our BNE.

- If there were N players, the BNE would be:

$$B(v_i) = \frac{N-1}{N} v_i.$$

- Strategically, the Dutch auction is equivalent to the first price auction.

- Summary Items on Auctions.

- (1) First price sealed bid auction: shade your bid to $(n-1)/n * v_i$.
- (2) Second price sealed bid auction: bid your valuation.
- (3) English auction: bid up to your valuation.
- (4) In common value auctions, the item up for auction has a common value V, but each bidder only has a noisy estimate of V, say $V_i + \epsilon_i$, where $\epsilon_i \sim (0, \sigma^2)$. The winner is the bidder with the highest error term!
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$$w(e_p) = qy(H, e_p) + (1-q)y(L, e_p),$$

• Separating Equilibrium The H type worker chooses $e = e_s$ and the L type worker chooses $e = e_L$.

$$w(e) = \begin{cases} y(H, e), & e \geq e_s \\ y(L, e) & e < e_s \end{cases}$$

• Hybrid or Partially Pooling Equilibria: The H type worker chooses $e = e_h$ always and the L type worker chooses $e = e_h$ with probability π and $e = e_l$ with probability $1 - \pi$. The firm's beliefs after observing e_h must be updated using Bayes rule.

$$w(e) = \begin{cases} q'y(H, e_h) + (1-q')y(L, e_h), & e \geq e_h \\ y(L, e_L) & e < e_h \end{cases}$$

29.5 Notes on Problem Sets and Exams

- In monopoly:

$$\pi_1 \geq \pi_2 \geq \pi_d \geq \pi_m.$$

$$\pi_1 \geq \pi_3 \geq \pi_m.$$

- Nash existance theorem may fail with an infinite strategy set.

- If you mix, the strategies must yield the same expected payoff (and if there is a third, it must be strictly less).

- A strictly dominate strategy is a strategy that yields a strictly higher payoff to the player NO MATTER what anyone else is doing.

- Always check for mixed strategies ... mixed NE can be used as triggers for Folk region payoffs. We can also mix between NE as a trigger to attain an even larger set of Folk payoffs.