# Economics 601: Macroeconomics
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Updated: January 1, 2005

# 1 Lecture 1: August 30, 2004

## 1.1 Consumption: Theory and Evidence

- The classic Keynesian consumption function is of the form:

$$c = A + (M.P.C) * y.$$

  Where $c$ is consumption, $A$ is autonomous consumption spending, $M.P.C.$ is the Marginal Propensity to Consume $(0, 1)$, and $y$ is household income. Note this is all at one time period so consumption today depends only on income today (MYOPIC consumers).

- Modigliani had a criticism of this model and asked, what about retirement? Hence he developed the Lifecycle consumption model where consumers save to smooth consumption over a life time.

- Friedman also criticized the Keynesian model saying that some shocks to income are permanent and others are just transitory. In reality consumers know which is which and will not immediately increase consumption in response to a transitory income shock (a farmers crop prices rise in only one season). Hence he developed the Permanent Income Hypothesis. Thus consumption should be a function of permanent income.

- In reality, both Modigliani and Friedman had the same criticism of the Keynesian model: consumers are not myopic and in actuality, they are forward looking.

## 1.2 The Life Cycle/Permanent Income Hypothesis

- Key assumptions: 1) Dynamic Optimization: forward looking behavior. 2) Perfect Credit Markets: households can borrow or save at the market interest rate, $r$, subject only to a solvency constraints (you can not overspend the present discounted value (pdv) of your lifetime income).

- Example: 2 period problem under certainty. Preferences:

$$u(c_1) + \beta u(c_2), \quad \beta \in (0, 1), \quad u' > 0, \quad u'' < 0.$$

  Constraints:

$$c_1 = y_1 - s_1, \quad c_2 = (1 + r)s_1 + y_2.$$

  Where $y_1, y_2$ are labor or tranfer income assumed to be non-negative and do NOT include asset income. Savings, $s_1$, can be negative, positive or 0. Combining the constraints yield:

$$c_1 + \frac{c_2}{1 + r} = y_1 + \frac{y_2}{1 + r} \equiv h_0.$$

  This is the Intertemporal Budget Constraint (IBC) and says that the pdv of consumption must equal the pdv of lifetime income.

- Thus the maximization problem becomes:

$$max_{c_1,c_2} \;\; L = u(c_1) + \beta u(c_2) + \lambda(h_0 - c_1 - \frac{c_2}{1+r}).$$

  Where $\lambda$ is the lagrange multiplier or the value of relaxing the IBC by 1 unit of income in pdv terms in period 1 (so maybe $h_0$ increases by 1 unit).

- A necessary condition for an optimum is the FOC of the lagrangian ($L$) with respect to the control variables ($c_1, c_2$) must be 0.

$$c_1 : u'(c_1) = \lambda.$$

$$c_2 : u'(c_2) = \frac{\lambda}{\beta(1+r)}.$$

  Solve for $\lambda$ and set equal yields the following EULER equation:

$$u'(c_1) = \beta(1+r)u'(c_2).$$

  This is the optimality condition involving an intertemporal choice.

- Note that $u'(c_1)$ is the value of consuming a marginal unit of income in period 1. $\beta(1+r)u'(c_2)$ is the value of using a marginal unit of income to increase savings in period 1, and consuming the accumulated proceeds in period 2. Along the optimal path, this equality must hold.

- Note that if $\beta(1+r) = 1$, then $u'(c_1) = u'(c_2) \implies c_1 = c_2$. This is called CONSUMPTION SMOOTHING. And the intuition comes from the fact that $u$ is concave. Hence :

$$u(x) > 0.5u(x-\epsilon) + 0.5u(x+\epsilon).$$

  See [G 1.1] in notes. Consumers do better by smoothing income (even keel) then by consuming a lot in one period and less in the other.

- If $\beta(1+r) < 1$, then:

$$u'(c_1) < u'(c_2) \implies c_1 > c_2.$$

- If $\beta(1+r) > 1$, then:

$$u'(c_1) > u'(c_2) \implies c_1 < c_2.$$

- In general, the consumption growth, $(\frac{c_2}{c_1})$ is an increasing function of the interest rate along the optimal path. The intuition comes from INTERTEMPORAL SUBSTITUTION: High interest rates implies that the relative price of $c_2$ has fallen because fewer units of forgone $c_1$ are requred to obtain a unit of $c_2$.

- So we have 2 motives which cause tension: Consumption Smoothing and Intertemporal Substitution. Note that the timing of consumption is independent of the timing of income ($y$) along the optimal path under certainty ($y_1, y_2$ are not in the euler equation). This is completely the opposite of Keynes.

- $$\text{If } y_1 << y_2 \Rightarrow \text{Borrow to increase } c_1.$$
$$\text{If } y_1 >> y_2 \Rightarrow \text{Save to increase } c_2.$$

- Now consider the specialied utility function of the form:
$$u(c) = \frac{\sigma}{\sigma - 1} c^{(\sigma-1)/\sigma}, \quad \sigma > 0, \sigma \neq 1.$$
$$u(c) = ln(c), \quad \sigma = 1.$$
This is called the POWER utility function, the Constant Elasticity of Substitution (CES) utility function, or the Constant Relative Risk Aversion (CRRA) utility function.

- Plugging this into the euler equation yields:
$$\frac{c_2}{c_1} = [\beta(1+r)]^{\sigma}.$$
Note that given $\beta(1+r)$, the slope of consumption growth depends on $\sigma$. See the graphs in notes [G-1.2]. $\sigma$ is the Intertemporal Elasticity of Substitution (IES). This follows from:
$$\frac{d(ln(c_2/c_2))}{d(ln(1+r))} = \sigma,$$
which is constant. $\sigma$ is also independent of the levels of consumption. This a very particular property and usually not the case (the rich and the poor will respond the same to a change in $r$ for example). Intuition: Why is $\sigma$ the IES? Because $\sigma$ governs the curvature of the utility function. See [G-1.3] in notes.

- If $\sigma \to \infty$, $\dfrac{\sigma - 1}{\sigma} \to 1$ and $u$ asymptotically approaches a linear function. Under a linear utility function, there is no impetus for consumption smoothing. The goal is to maximize the pdv of lifetime consumption so the consumer will pursue intertemporal substitution opportunities to the maximum extent.

- If $\sigma$ is lower, then the utility function is highly curved and the smoothing motive is stronger. The consumer will pursue intertemporal substitution opportunities less strongly so there will be less tilting of the consumption path (more even keeled).

# 2 Lecture 2: September 1, 2004

- Recall from last time we had the simple 2 period, no uncertainty problem with Power/CES utility function:

$$u(c) = \frac{\sigma}{\sigma - 1} c^{(\sigma-1/)\sigma}, \quad \sigma \neq 1,$$
$$u(c) = ln(c), \quad \sigma = 1.$$

- We found the euler equation:

$$u'(c_1) = \beta(1+r)u'(c_2).$$

- And substitution yields:

$$\frac{c_2}{c_1} = [\beta(1+r)]^{\sigma}.$$

- Now combine this with the intertemporal budget constraint (IBC):

$$c_1 + \frac{c_2}{1+r} = y_1 + \frac{y_2}{1+r} \equiv h_0.$$

Solve for $c_2$:

$$c_2 = (1+r)(h_0 - c_1).$$

Substitute:

$$c_1 = (1+r)(h_0 - c_1)[\beta(1+r)]^{-\sigma}.$$
$$c_1(1 + \beta^{-\sigma}(1+r)^{1-\sigma}) = h_0\beta^{-\sigma}(1+r)^{1-\sigma}.$$
$$c_1 = \frac{h_0\beta^{-\sigma}(1+r)^{1-\sigma}}{1 + \beta^{-\sigma}(1+r)^{1-\sigma}}.$$
$$c_1 = \frac{h_0(1+r)^{1-\sigma}}{\beta^{\sigma} + (1+r)^{1-\sigma}}.$$
$$c_1 = \frac{h_0}{\beta^{\sigma}(1+r)^{\sigma-1} + 1}.$$

And if $\sigma = 1$,

$$c_1 = \frac{h_0}{1+\beta}.$$

- In general, for a 2 period model under certainty,

$$IES = \frac{d\, ln(\frac{c_2}{c_1})}{d\, ln(1+r)}.$$

For a CES utility function, the $IES = \sigma$ always. This is independent of levels of consumption and $h_0$. Usually this is not the case. Why does $\sigma$ govern the IES? We have two competing forces:

- Intertemporal Substitution: consume more in periods when $c$ is cheap. If $r$ is high, defer lots of $c$ until the future.
- Consumption Smoothing: Keep $c$ as constant as possible in all periods due to the concavity of $u(\cdot)$. The strength of this motive depends on the curvature of the utility function. If $\sigma$ is high, $u(\cdot)$ is almost linear so there is a weak consumption smoothing motive.

- So with this in mind, we ask the question: "Do higher interest rates increase saving?" Consider:

$$s_1 = y_1 - c_1.$$

$$s_1 = y_1 - c_2[\beta(1+r)]^{-\sigma}.$$

Thus,

$$\frac{ds_1}{dr} = -\frac{dc_1}{dr}.$$

For the solution for $c_1$ above, we have two cases:

  - Case 1: $y_2 = 0$. Then:

$$\frac{ds_1}{dr} \gtreqless 0 \; as \; \sigma \gtreqless 1.$$

  This is basically just the income and substitution effects coming into play. As $r$ increases, the optimal $\frac{c_2}{c_1}$ also rises via the substitution effect (and also can be seen from our Euler equation). But the income effect on $y_1$ says that as $r$ rises, the lifetime opportunity set expands, and since $c_1$ is a normal good, this tends to increase $c_1$. Note that if $\sigma = 1, u(c) = ln(c)$ and there is no impact because $c_1$ does not depend on $r$.

  - Case 2: $y_2 > 0$ and $h_0 = y_1 + \frac{y_2}{1+r}$. Then:

$$\frac{d\,h_0}{dr} = \frac{-y_2}{(1+r)^2} < 0.$$

  Since $c_1$ is increasing in $h_0$, this provides an additional channel by which increasing the interest rate will lead to more saving. The income effect is now also on $y_2$ and it depresses $c_1$. In log case, $\frac{ds_1}{dr} > 0$ as long as $y_2 > 0$.

- Empirical Testing. If the world was certain, and we had data on $c$ and $y$ over time for households, how could we test the lifecycle hypothesis (LCH)? In general, we would like to test the implications of models that are not shared by plausible alternatives.

- We could for example test whether people with higher lifetime $h_0$ have a higher lifetime $c$. But this wouldn't give us much because most models will predict this implication. Test would lack power.

- A better idea would be to test whether the timing of consumption over the lifecycle is correlated with the timing of income over the lifecycle. In the LCH, there should be no correlation between $\Delta c$ and $\Delta y$ under certainty. This implication is not shared by the other leading alternatives. For example, in the Keynesian model:

$$\Delta c = M.P.C. * \Delta y.$$

This is of course in sharp contrast to the LCH. Another model we could test is the liquidity constrained model developed next.

## 2.1 Liquidity Constrained Model

- Consider a model where households are rational and forward looking but credit markets are imperfect and consumers cannot borrow. Thus we have a new constraint:

$$s_1 \geq 0 \Longleftrightarrow c_1 \leq y_1.$$

- Set up the lagrangian:

$$L = u(c_1) + \beta u(c_2) + \lambda(h_0 - c_1 - \frac{c_2}{1+r}) + \pi(y_1 - c_1).$$

$\pi$ is the shadow value of relaxing the borrowing constraint, holding $h_0$ constant (the other constraint). This is equivalent to saying that $\pi$ is the shadow value of reducing $y_2$ by $(1+r)$ units and increasing $y_1$ by 1 unit.

- FOCs:

$$c_1 : u'(c_1) = \lambda + \pi.$$

$$c_2 : \beta u'(c_2) = \frac{\lambda}{1+r}.$$

Complementary Slackness (on the inequality constraint):

$$\pi(y_1 - c_1) = 0.$$

So either the constraint binds ($y_1 = c_1$) or the shadow price is 0.

- Euler equation:

$$u'(c_1) = \beta(1+r)u'(c_2) + \pi.$$

Or,

$$u'(c_1) \geq \beta(1+r)u'(c_2).$$

We we can say that the agent will never, at an optimum, over-consumer in period 1. $u'(c_1)$ will never be too low. However, if the liquidity constraint (L.C.) binds, it is possible for $u'(c_1)$ being "too high" due to $c_1$ being "too low". Though the agent will not be able to transfer consumption from period 2 to 1 because of the L.C. The agent cannot borrow.

- In this simple example, the closed form solution is as follows: Let $c_1^*$ be the optimal $c_1$ without the L.C. Then define 2 groups:

- Group 1: If $c_1^* \leq y_1$, then the optimal $c_1$ under the L.C. is $c_1^*$, $\pi = 0$, L.C. does not bind.
- Group 2: If $c_1^* > y_1$, then the optimal $c_1$ will be $y_1$ under the L.C., $\pi > 0$, and the L.C. binds.

- So for people in Group 1, the LCH Euler equation holds with equality and $\Delta c$ is uncorrelated with $\Delta y$. In Group 2, the LCH does not hold so $\Delta c$ is correlated with $\Delta y$ because $c_1 = y_1$, $c_2 = y_2$, which is like the Keynesian case with $M.P.C. = 1$. Group 1 people tend to be people with a relatively high $\frac{y_1}{y_2}$ so they are likely to be unconstrained. These could be people with high initial wealth endowments or declining income paths. In Group 2, we have the poor graduate students with a lot to gain in the future but are completely constrained today. So $\frac{y_1}{y_2}$ is relatively low. We face binding L.C.'s.

## 2.2 Uncertainty

- We need to describe ex-ante preferences over random commodity bundles, $c$. Assume $c$ has a conditional probability distribution, $\pi(c)$. $\pi(c)$ is conditional on current information and choices. We also assume that people's subjective conditional distribution, $\pi(c)$, is correct – Rational Expectations.

- So with atemporal preferences, define the EXPECTED utility function as:

$$E[u(c)] = \int u(c)\pi(c)dc.$$

Where $u(\cdot)$ is a standard atemporal utility function which is concave and increasing over non-random commodity bundles.

- EXAMPLE. Suppose $c = c(x)$ where $x$ is a discrete random variable with $N$ possible realizations: $(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n)$. This could be income for example. Each realization has probability $\pi_i$ of being realized with $\pi_i \geq 0 \; \forall \; i$ and $\sum_i \pi_i = 1$. Then:

$$E[u(c)] = \sum_{i=1}^{N} \pi_i u(c(\bar{x}_i)) = \sum_{i=1}^{N} \pi_i u(\bar{c}_i).$$

- Consider a simple example of 2 lotteries. Let $u(c) = log(c)$. Suppose our choice is either:

$$\text{Lottery 1: } C = H \text{ with probability } = \frac{1}{2}, \;\; C = L \text{ with probability } = \frac{1}{2}.$$

$$\text{Lottery 2: } C = \frac{L+H}{2} \text{ with probability } = \; 1.$$

The expected utility from lottery 1:

$$E[u] = 0.5log(L) + 0.5log(H).$$

The expected utility from lottery 2:

$$E[u] = log(\frac{L+H}{2}).$$

- Note that $u(\cdot)$ is concave, we thus choose lottery 2. In fact for any concave utility function,

  $$E[u(c)] \leq u[E(c)].$$

  Attaining average consumption with certainty is always better. This is your typical risk aversion principal. See graph in notes [G-2.1].

# 3 Lecture 3: September 8, 2004

## 3.1 Decision Making Under Uncertainty

- In a 1-period model, the household has choice over random consumption lotteries. Each has some distribution $\pi(c)$ known to the households.

- Usually we represent preferences using the Expected Utility Function:

$$E[u(c)] = \int u(c)\pi(c)dc, \quad or \quad \sum_{i=1}^{N} u(c_i)\pi_i.$$

- One result from all of this is Risk Aversion: If $u''(\cdot) < 0$, concave, then:

$$E[u(c)] < u[E(c)].$$

So the consumer prefers the sure thing which is the average of the lottery payoffs to the risk of the lottery.

- We quanitfy risk aversion in the following way. Suppose in a 1-period example, baseline consumption is $c$, and there is an additive risk equal to $\epsilon$. Suppose WLOG:

$$\epsilon \sim (0, \sigma_\epsilon^2).$$

Define the risk premium as the $p$ that satisfies:

$$\underbrace{E[u(c+\epsilon)]}_{No\ Insurance} = \underbrace{u(c + E[\epsilon] - p)}_{Insured}.$$

Thus $p$ is the maximum willingness to pay to insure yourself against the risk, $\epsilon$.

- **Theorem**: (Pratt 1964) For small enough additive risks,

$$p \approxeq \frac{1}{2}\sigma_\epsilon^2 \left( -\frac{u''(c)}{u'(c)} \right).$$

Where the term in parens is the "Coefficient of Absolute Risk Aversion" or ARA. Notice that it varies with $c$.

- Now suppose the risk term, $\epsilon$, is multiplicative. Define $p$ as the fraction of baseline consumption you would be willing to pay to fully insure yourself again the $\epsilon$ risk:

$$E[u(c\epsilon)] = u[cE(\epsilon) - pc].$$

WLOG, assume:

$$\epsilon \sim (1, \sigma_\epsilon^2).$$

- **Theorem**: (Pratt 1964) For small enough multiplicative risks,

$$p \approxeq \frac{1}{2}\sigma_\epsilon^2\left( - \frac{u''(c)c}{u'(c)}\right).$$

Where the term in parens is the "Coefficient of Relative Risk Aversion" or RRA. Notice that it varies with $c$ and also has an extra term in the numerator.

- Thus the coefficient of relative risk aversion is the absolute value of the elasticity of $u'(c)$ wrt $c$. It's basically saying how fast does the marginal utility of consumption fall as consumption rises.

- Consider the power utility function:

$$u(c) = \frac{\sigma}{\sigma - 1}c^{(\sigma-1)/\sigma}.$$

Then the coefficient of absolute risk aversion is as follows:

$$ARA = \frac{-u''(c)}{u'(c)} = \frac{\frac{1}{\sigma}c^{(-1-\sigma)/\sigma}}{c^{-1/\sigma}} = \frac{1}{\sigma c}.$$

Then,

$$RRA = \frac{-u''(c)c}{u'(c)} = c * ARA = \frac{1}{\sigma}.$$

## 3.2 Multiple Period Models

- Households choose lotteries over sequences of consumption over time.

- The most common way to represent preferences is the TIME SEPARABLE, ADDITIVE, EXPECTED UTILITY FUNCTION:

$$E_0\left[\sum_{t=0}^{T}\beta^t u(c_t)\right].$$

Note the expectation is taken wrt to the conditional distribution over future $\{c_t\}_{t=0}^T$.

- As long as preferences are of this form, we have:

$$IES = \frac{1}{RRA},$$

for any $u(c)$, at any baseline $c$.

- In the case of POWER utility,

$$IES = \sigma.$$

$$RRA = \frac{1}{\sigma}.$$

- Alternative preference formulations break the link between IES and the coefficient of RRA because some economists think this is unrealistic. See Kreps-Porteus/Epstein-Zin or Phillipe Weil.
- The intuition for the link is that both the risk aversion motive and consumption smoothing motive depend on the degree of curvature of the utility function. See graph [G-3.1] in notes.
- For low curvature in the utility function (e.g. a high $\sigma$ in the Power utility function) implies both WEAK consumption smoothing over time (High IES) AND a willingness to bear risks. So we have a willingness to tolerate consumption variation over TIME and STATES of NATURE.
- For high curvature in the utility function (e.g. a low $\sigma$ in the Power utility function) implies both strong consumption smoothing over time (low IES) AND an aversion to risks. So we have an unwillingness for consumption variation over TIME and STATES of NATURE.

## 3.3 Representing Repeated Uncertainty

- Suppose there are $T$ periods, $t = 1...T$ and each period there exists a random varible $x_t$ with distribution $\pi_t(x_t)$. $x_t$ might be labor income realizations.
- **Definition** A HISTORY, $x^J$, contains all realizations of $x_t$ through the period $J$. Thus:
$$x^J = \{x_1, x_2, x_3, \ldots, x_J\}.$$
Note that $x^J$ is itself a composite random variable with dimension $J$. We say that $x^J$ has a probability distribution:
$$\pi^J(x^J).$$
- Example. If $\pi_t$ has $N$ realizations in each period $t$, then $x^J$ will have $N^J$ possible realizations (think of a tree latice).
- In general, $C_J$ will depend on the entire $x^J$: $c_t = c_t(x^t)$. So in the discrete case, we can write expected utility using this History notation as the following:
$$E_0 \sum_{t=0}^{T} \beta^t u(c_t) = \sum_{t=0}^{T} \sum_{x^t}^{N^t} \beta^t u[c_t(x^t)]\pi_0^t(x^t).$$
Note that the superscripts on $\beta$ and $N$ are powers while the rest represent histories. Bad notation.
- And the question now becomes: How does $\pi^J(x^J)$ relate to the underlying primitives: $\pi_t(x_t)$?
- Example 1: IID Uncertainty. $\pi_t(x_t) = \pi(x_t)$. Time Invariant. This implies:
$$\pi^J(x^J) = \pi(x_1)\pi(x_2)\cdots\pi(x_J).$$

- Example 2: Markov Uncertainty. Recall that $x_t$ has the markov property if:

$$\pi_{t+1}(x_{t+1}|x^t) = \pi(x_{t+1}|x_t).$$

So $x_t$ is a sufficient statistic for $x^t$ (the entire history of $x$ at period $t$ only depends on the last time period, $t-1$). We might say the conditional distribution of $x_{t+1}$ exhibits "One-Period Persistance" or is "First Order Autoregressive." This implies:

$$\pi^J(x^J) = \pi(x_0)\pi(x_1|x_0)\pi(x_2|x_1)\cdots\pi(x_J|x_{J-1}).$$

- Example 3: Markov Chain. Let $x_t$ be discrete with $N$ possible realizations in each period. Then we can represent $\pi(x_{t+1}|x_t)$ using a markov transition matrix $p$, which is $NxN$, such that:

$$p(i,j) = \pi(x_{t+1} = \overline{x}_j | x_t = \overline{x}_i).$$

Where $p(i,j)$ is the $i^{th}$ row, $j^{th}$ column of $p$. This means the rows of $p$ sum to 1. It says what is the probability that $x$ is equal to $\overline{x}_i$ today and will be equal to $\overline{x}_j$ tomorrow. This implies:

$$\pi^J(x^J) = \pi(x_0)p(x_0,x_1)p(x_1,x_2)\cdots p(x_{J-1},x_J).$$

This is a slight abuse of notation.

## 3.4 The Lifecycle Hypothesis/Permanent Income Hypothesis Under Uncertainty

- Probem:

$$max\left\{E_0\sum_{t=0}^{T}\beta^t u(c_t)\right\},$$

Subject to:

$$(1)\ \ a_0\ given,$$

$$(2)\ \ a_{t+1} = (a_t + y_t - c_t)R_{t+1},$$

$$(3)\ \ R_{t+1} = \phi_t(1+z_{t+1}) + (1-\phi_t)(1+r_{t+1}),$$

$$(4a)\ \ a_{T+1} = 0 \text{ if } T \text{ is finite},$$

$$(4b)\ \ lim_{L\to\infty} a_L\Big[\prod_{t=1}^{L}\Big(\frac{1}{1+r_t}\Big)\Big] = 0 \text{ if } T \text{ is infinite}.$$

- Definitions:
  - $a_t$: financial wealth at the beginning of period $t$ (could be positive or negative).
  - $y_t$: flow of labor income in period $t$, known in period $t$ but may be unknown in period $t-1$.
  - $R_{t+1}$: portfolio return on savings between periods $t$ and $t+1$.

- $\phi_t$: share of portfolio held in risky asset, not bound between zero and one since short selling is permitted.
- $z_{t+1}$: rate of return on your risky asset (Stocks), known as of $t+1$ but unknown in period $t$.
- $r_{t+1}$: risk free rate, known in period $t$ though my vary over time.

- Note that $R_{t+1}$ is uncertain in period $t$ if $\phi_t \neq 0$.
- Constraint 4 is just the No-Ponzi game criteria.

# 4 Lecture 4: September 13, 2004

## 4.1 Lifecycle Model/Permanent Income Hypothesis Under Uncertainty

- Consider again the consumer's problem:

$$max\ E_0[\sum_{t=0}^{T}\beta^t u(c_t)],$$

Subject to:

$$a_0\ given,$$

$$a_{t+1} = (\underbrace{a_t + y_t}_{Cash\ On\ Hand} - c_t)R_{t+1},\ \text{DBC:Dynamic Budget Constraint}$$

$$R_{t+1} = (1 + r_{t+1}) + \phi_t(\underbrace{z_{t+1} - r_{t+1}}_{Excess\ Return\ on\ Risky\ Asset}),$$

$$a_{T+1} = 0\ \text{ Solvency condition if T is finite,}$$

$$lim_{L\to\infty} a_L[\prod_{t=1}^{L}\frac{1}{1+r_t}] = 0\ \text{ Solvency condition if T is infinite.}$$

- Note that the solvency condition in the finite case may require that $c_T < 0$.
- Note also the the solvency condition in the infinite case does NOT say that:

$$lim_{L\to\infty} a_L = 0.$$

All it says is that debt must grow slower than the interest rate in the limit. No Ponzi Games (NPG). An example of a violation of NPG is $a_T < 0$ and $c_{T+k} = y_{T+k}\ \forall\ k \geq 0$.

- The solution to this problem could involve Dynamic Programming. The first step is defining choice and state variables.
- Choice Variables: $c_t$, $\phi_t$. The agent has FULL control over these.
- State Variables: Predetermined at time $t$ and they affect current and/or future opportunities or utility.
  - Endogeneous State Variables: $a_t$. Current choices effect future states.
  - Exogeneous State Variables: $y_t$, $r_{t+1}$, $T - t$ (remaining lifespan), the probability distribution of $z_{t+1}$ (since $z_{t+1}$ itself is unknown at time $t$), future values of $y$, $z$, and $r$, and finally the probability distribution of $z_{t+1}$ could be a function of the histories: $y^t$, $r^{t+1}$, and $z^t$.
- Note the we should always use the MINIMUM number of sufficient state variables. For example $z_t$ is not listed as a state variable because if you know $a_t$, $z_t$ is implied. Do NOT list redundant states.

- What does the household choose at time 0? Under certainty: the entire sequence, $\{c_t\}_{t=0}^T$ should be completely decided and hence we have non-stochastic consumptions. Under uncertainty, this is neither feasible or optimal. Instead, at time 0, the household chooses $c_0, \phi_0$ and a decision rule mapping all possible future realizations of state variables into future choice variables $\{(c_t, \phi_t)\}_0^T$. "We can leave this decision rule to the lawyers."

- Let $\{c_t^*, \phi_t^*\}_{t=0}^T$ be the random sequence of optimal choices implied by this decision rule (DR). We claim that the DR is TIME CONSISTANT. That is, if the agent could reoptimize in period 1, the agent would choose same DR.

- A counter example to Time Consistancy is Hyperbolic Discounting (Laibson). Lifetime utility is as follows:

$$u(c_0) + \alpha E_0 \sum_{t=1}^{T} \beta^t u(c_t), \quad 0 < \alpha \leq 1.$$

  If we have $\alpha = 1$, we have exponential discounting which is the same as our original problem. If $\alpha < 1$, there is some additional discounting. In particular, the discount factor for comparing:

| | $c_0$ vs $c_1$ | $c_1$ vs $c_2$ |
|---|---|---|
| time 0 self | $\alpha\beta$ | $\dfrac{\alpha\beta^2}{\alpha\beta} = \beta$ |
| time 1 self | irrelevant | $\alpha\beta$ |

  With hyperbolic discounting ($\alpha < 1$), the time 1 self will generally tend to prefer a higher $\dfrac{c_1}{c_2}$ than the time 0 self finds optimal. Time 1 self discounts $c_2$ more.

## 4.2 Value Function

- Define the value function as:

$$V_t = max\ E_t \left[ \sum_{s=t}^{T} \beta^{s-t} u(c_s) \right],$$

  subject to constraints. It says how good it feels to be alive today. In general $V_t$ is a function of all state variables at time $t$. I will write $V_t = V_t(a_t)$, as function of endogeneous state variables where the time subscript on the $V$ implicitly includes all the other (possibly exogeneous) state variables.

- Properties of the Value Function.

$$(1)\ \ V_0(a_0) = u(c_0^*) + E_0[\sum_{t=1}^{T} \beta^t u(c_t^*)].$$

$$(2)\ \ V_1(a_1) = E_1[\sum_{t=1}^{T} \beta^{t-1} u(c_t^*)].$$

Note the information set in period 1 may differ from period 0, but because of time consistancy, $c_t^*$ is the same random variable in (1) and (2).

- **Fact:** Law of Iterated Expectations. For any random variable, $x$, and for any $k > 0$,

$$(3)\ \ E_t[E_{t+k}[x]] = E_t[x].$$

Our best forecast today of our best forecast tomorrow must be equal to our best forecast today.

- Apply (3) to (2) and multiply both sides by $\beta$,

$$(4)\ \ \beta E_0 V_1(a_1) = \beta E_0 E_1[\sum_{t=1}^{T} \beta^{t-1} u(c_t^*)] = E_0[\sum_{t=1}^{T} \beta^{t} u(c_t^*)].$$

- Combine (1) and (4):

$$(5)\ \ V_0(a_0) = u(c_0^*) + E_0[\sum_{t=1}^{T} \beta^{t} u(c_t^*)] = u(c_0^*) + \beta E_0 V_1(a_1).$$

- Evaluted at a general time $t$:

$$(6)\ \ V_t(a_t) = max_{(c_t, \phi_t)} \left\{ u(c_t) + \beta E_t V_{t+1}(a_{t+1}) \right\},$$

subject to:

$$a_{t+1} = (a_t + y_t - c_t) R_{t+1}.$$

- Equation (6) is the Bellman's Equation for the original dynamic programming problem. Bellman's Principal of Optimality says we can represent the full dynamic programming problem as a sequence of recursive 2 period problems. Note this is NOT myopic because $V_{t+1}$ embodies all periods beyond $t + 1$. Note that also, in general:

$$V_t \neq max\ u(c_t) + \beta E_t u(c_{t+1}).$$

- So what can we do with the Bellman's Equation? If we knew $V_{t+1}$, we could solve as a 2 period problem. In general, we are not given the form of the value function, $V_{t+1}$. However, 3 points:
  - 1) In some cases, we can guess and verify the correct $V(a)$.
  - 2) We can use repeated backwards recursion to solve for the $V$'s computationally. If the horizon is infinite, under certain other assumptions, $V_t(a)$ will be a time-

invariant function $V(a, states)$ so the computer can be instructed to do backwards induction until the value function converges.

– 3) Finally, we can learn about properties of the solution even without ever solving for $V$. The Euler equations can be found. Next time...

# 5 Lecture 5: September 15, 2004

## 5.1 Solving the Bellman's Equation in the LCH/PIH Under Uncertainty

- Recall the problem:

$$max\ E_0[\sum_{t=0}^{T}\beta^t u(c_t)],$$

Subject to:

$$a_{t+1} = (a_t + y_t - c_t)R_{t+1},$$
$$R_{t+1} = (1 + r_{t+1}) + \phi_t(z_{t+1} - r_{t+1}),$$

with the initial wealth given and solvency conditions. The bellman's equation is therefore:

$$V_t(a_t) = max_{(c_t,\phi_t)}\left\{u(c_t) + \beta E_t V_{t+1}(a_{t+1})\right\},$$

subject to:

$$a_{t+1} = (a_t + y_t - c_t)R_{t+1}.$$

Substituting,

$$V_t(a_t) = max_{(c_t,\phi_t)}\left\{u(c_t) + \beta E_t V_{t+1}\overbrace{[(a_t + y_t - c_t)\underbrace{[(1 + r_{t+1}) + \phi_t(z_{t+1} - r_{t+1})]}_{R_{t+1}}]}^{a_{t+1}}\right\},$$

Remember that although we write $V_t$ as an explicit function of only $a_t$, it is also an implicit function (through the subscript $t$) of other states.

- FOCs:

$$c_t:\quad u'(c_t) = \beta E_t[R_{t+1}V'_{t+1}(a_{t+1})]\quad (1).$$

So at the optimum, the benefit of consuming the marginal unit today must equal the discounted expected benefit of having more wealth tomorrow. Note this assumes we put the marginal unit of savings into the "OPTIMAL" portfolio.

$$\phi_t:\quad \beta(a_t + y_t - c_t)E_t\Big[[z_{t+1} - r_{t+1}]V'_{t+1}(a_{t+1})\Big] = 0\quad (2).$$

This has less of an intuitive interpretation. Assume $(a_t + y_t - c_t) \neq 0$ so the agent is actually investing in something. Then:

$$E_t\Big[[z_{t+1} - r_{t+1}]V'_{t+1}(a_{t+1})\Big] = 0\quad (3).$$

$$E_t\Big[[z_{t+1} - r_{t+1} + 1 - 1]V'_{t+1}(a_{t+1})\Big] = 0.$$

$$E_t(z_{t+1} + 1)V'_{t+1}(a_{t+1}) - E_t(r_{t+1} + 1)V'_{t+1}(a_{t+1}) = 0.$$

$$E_t(z_{t+1}+1)V'_{t+1}(a_{t+1}) = E_t(r_{t+1}+1)V'_{t+1}(a_{t+1}).$$
$$E_t(z_{t+1}+1)V'_{t+1}(a_{t+1}) = (r_{t+1}+1)E_tV'_{t+1}(a_{t+1}).$$

Or:

$$(1+r_{t+1})E_tV'_{t+1}(a_{t+1}) = E_t(1+z_{t+1})V'_{t+1}(a_{t+1}). \quad (4).$$

Which says the expected benefit of putting the marginal unit into the riskless asset equals the expected benefit of putting the marginal unit into the risky asset.

- Notice the RHS of (4) is NOT equal to $E_t(1+z_t) * E_tV'_{t+1}(a_{t+1})$. There would have to be a covariance term there. Thus (4) DOES NOT say:

$$(1+r_{t+1}) = E_t(1+z_{t+1}).$$

  This would say that the expected value of the risky return equaled the riskless return which isn't realistic. The covariance term embodies the risk premium.

- Plug (3) into (1) yields:
  Start with (1) and expand $R_{t+1}$:

$$u'(c_t) = \beta E_t[((1+r_{t+1})+\phi_t(z_{t+1}-r_{t+1}))V'_{t+1}(a_{t+1})]$$

  Multiply through the expectation:

$$u'(c_t) = \beta E_t[V'_{t+1}(a_{t+1})(1+r_{t+1})] + \beta\phi_t \underbrace{E_t[V'_{t+1}(a_{t+1})(z_{t+1}-r_{t+1})]}_{0\ by\ (4)}$$

$$u'(c_t) = \beta E_t[V'_{t+1}(a_{t+1})(1+r_{t+1})]$$

  $r_{t+1}$ is known at time $t$ so pull it out of the expectation:

$$u'(c_t) = \beta(1+r_{t+1})E_tV'_{t+1}(a_{t+1}). \quad (5).$$

  So the expected benefit of putting the marginal unit of wealth into the riskless asset must equal the expected benefit of consumption today.

- By the optimal portfolio choice, at the margin, we are indifferent between allocating the extra unit of savings to the riskless asset, the risky asset, or any linear combination of the two. Note we could have also wrote:

$$u'(c_t) = \beta E_t[(1+z_{t+1})V'_{t+1}(a_{t+1})]. \quad (5').$$

  This is a nice (and convenient) result: We can evaluate the marginal benefit of saving by looking at either asset individually.

- So a Problem Remains: What is $V$ ?? We don't know the specific form of the value function, $V$, but what appears in equations (1) - (5) is actually $V'(a)$ and not $V(a)$

itself. Thus, in principal:

$$V_t'(a_t) = \frac{dV_t}{da_t} = \underbrace{\frac{\partial V_t}{\partial a_t}}_{Direct\ Effect} + \underbrace{\frac{\partial V_t}{\partial c_t}\frac{\partial c_t}{\partial a_t} + \frac{\partial V_t}{\partial \phi_t}\frac{\partial \phi_t}{\partial a_t}}_{Indirect\ Effects}.$$

Since $V$ is a maximized value function, by the envelope theorem:

$$\frac{\partial V_t}{\partial c_t} = \frac{\partial V_t}{\partial \phi_t} = 0.$$

So we have:

$$V_t'(a_t) = \frac{dV_t}{da_t} = \underbrace{\frac{\partial V_t}{\partial a_t}}_{Direct\ Effect}.$$

Recall the form of the Bellman's Equation:

$$V_t(a_t) = max_{c_t,\phi_t}\Big\{u(c_t) + \beta E_t V_{t+1}[(a_t + y_t - c_t)R_{t+1}]\Big\}.$$

Differentiating:

$$V_t'(a_t) = \beta E_t R_{t+1} V_{t+1}'(a_{t+1}). \quad (6)$$

But note that the RHS of (6) is equal to the RHS of (1). So:

$$V_t'(a_t) = u'(c_t) \quad (7).$$

Or moving forward:

$$V_{t+1}'(a_{t+1}) = u'(c_{t+1}). \quad (8)$$

- So the intuition is that at an optimum, we should be indifferent between devoting a marginal unit of wealth into either savings or consumption.
- Plug (8) into (5):

  $$u'(c_t) = \beta(1 + r_{t+1})E_t u'(c_{t+1}), \quad (9)$$

  which is our usual Euler Equation. The intuition for (9) is that at an optimum, we equalize the marginal expected benefit of consuming today versus saving in the risk free (or equivalently: risky) asset and consuming tomorrow. The only difference from the model under certainty is the expectation.
- Notice that (9) is still not a closed form decision rule for consumption. Can we find decision rules?
  - Numerically? Yes, for a broad class of problems (though this is only approximate).
  - Analytically? Yes, but only in special cases.
- Consider one of these special cases: Quadratic Utility with No Risky Assets. Define utility:

  $$u(c_t) = c_t - \frac{a}{2}c_t^2, \;\; a > 0.$$

$$\phi_t = 0 \Longrightarrow R_{t+1} = (1 + r_{t+1}).$$
$$(1 + r_{t+1}) = 1 + r, \quad constant.$$
$$\beta(1 + r) = 1.$$

Thus the Euler equation above:

$$u'(c_t) = \beta(1 + r_{t+1})E_t u'(c_{t+1}).$$

$$u'(c_t) = E_t u'(c_{t+1}).$$
$$1 - ac_t = E_t[1 - ac_{t+1}].$$
$$c_t = E_t c_{t+1}.$$

This is ONLY TRUE FOR QUADRATIC UTILITY.

- The the law of iterative expectations:

  $$c_t = E_t c_{t+1} = E_t E_{t+1} c_{t+2} = E_t c_{t+2}.$$

  So,

  $$c_t = E_t c_s \quad \forall\, s > t.$$

  We might call this expected consumption smoothing. "Consume today as if we believe we can consume the same amount for the rest of our lives."

- But this is STILL not a decision rule. To get a DR, we need to derive an intertemporal budget constraint from our dynamic budget constraint and our solvency condition. DBC for period 1:

  $$a_1 = (1 + r)(a_0 + y_0 - c_0).$$
  $$a_2 = (1+r)(a_1+y_1-c_1) = (1+r)\underbrace{(y_1 - c_1)}_{Primary\ Surplus\ in\ 1} + (1+r)^2(\underbrace{y_0 - c_0}_{Primary\ Surplus\ in\ 0}) + (1+r)^2 a_0.$$
  $$\cdots$$
  $$a_{s+1} = (1 + r)^{s+1} a_0 + \sum_{t=0}^{s} (1 + r)^{s+1-t}(y_t - c_t).$$

  Divide both sides by $(1 + r)^{s+1}$ and rearrange,

  $$\sum_{t=0}^{s} (1 + r)^{-t} c_t = a_0 + \sum_{t=0}^{s} (1 + r)^{-t} y_t - \frac{a_{s+1}}{(1 + r)^{s+1}}.$$

- For the finite horizon, $s = T$ and $a_{T+1} = 0$. For infinite horizon, $lim_{L\to\infty} \dfrac{a_{L+1}}{(1 + r)^{L+1}} = 0$, so we have:

  $$\sum_{t=0}^{s} (1 + r)^{-t} c_t = a_0 + \sum_{t=0}^{s} (1 + r)^{-t} y_t.$$

Note, this is the REALIZED intertemporal budget constraint that must hold ex-post on all sample paths. Note that the RHS of the IBC is a random variable at time 0, but we know the IBC will hold in EXPECTATION ex-ante:

$$E_0\left[\sum_{t=0}^{s}(1+r)^{-t}c_t\right] = a_0 + E_0\left[\sum_{t=0}^{s}(1+r)^{-t}y_t\right] \equiv a_0 + h_0.$$

Where $a_0$ is financial wealth and $h_0$ is human wealth:

$$h_t = E_t\left[\sum_{s=0}^{\infty}(1+r)^{-s}y_{t+s}\right].$$

- So now combine the expected IBC with the quadratic utility euler equation to find a closed form solution. From Euler:

$$c_0 = E_t c_t \ \forall t.$$

So expected IBC becomes:

$$E_0\left[\sum_{t=0}^{s}(1+r)^{-t}c_0\right] = a_0 + E_0\left[\sum_{t=0}^{s}(1+r)^{-t}y_t\right].$$

$$c_0\sum_{t=0}^{s}(1+r)^{-t} = a_0 + h_0.$$

$$c_0 = \frac{a_0+h_0}{\sum_{t=0}^{s}(1+r)^{-t}}.$$

In the infinite horizon case, we can simplify even more. Note:

$$\sum_{i=1}^{\infty}x^i = \frac{1}{1-x}, \quad x\in(0,1).$$

So,

$$\sum_{t=0}^{\infty}(1+r)^{-t} = \frac{1}{1-(1/(1+r))} = \frac{1+r}{r}.$$

And the equation for $c_0$ becomes:

$$c_0 = \frac{r}{1+r}(a_0+h_0).$$

At any time period $t$:

$$c_t = \frac{r}{1+r}(a_t+h_t).$$

And this is a CLOSED FORM DECISION RULE! So set $c_t$ as an annuity payment on total lifetime wealth as of time $t$.

# 6 Lecture 6: September 20, 2004

## 6.1 Formalizing Friedman's Intuitive Arguments

- Returning the quadratic utility model with $\phi_t = 0$, $(1 + r_{t+1}) = (1 + r) = \frac{1}{\beta}$, $T = \infty$, we found the decision rule:

$$c_t = \frac{r}{1+r}(a_t + h_t),$$

with,

$$h_t = E_t \sum_{s=0}^{\infty} (1+r)^{-s} y_{t+s}.$$

- We can apply this formula to solve for savings: $s_t = ra_t + y_t - c_t$:

$$s_t = ra_t + y_t - \frac{r}{1+r}(a_t + h_t).$$

$$s_t = (r - \frac{r}{1+r})a_t + y_t - \frac{r}{1+r}h_t.$$

$$s_t = \frac{r^2}{1+r}a_t + \underbrace{y_t - \frac{r}{1+r}h_t}_{\text{Current Income Less Permanent Income}}.$$

Note that if $y_t > \frac{r}{1+r}h_t$, this implies that current income is greater than the annuity payment on the PDV of our lifetime labor income, so we should increase savings today. The opposite is also true.

- Now consider the change in consumption, $\Delta c_t = c_t - c_{t-1}$ :

$$\Delta c_t = \frac{r}{1+r}(a_t - a_{t-1} + h_t - h_{t-1}).$$

By the homework 3 solution:

$$\Delta c_t = \frac{r}{1+r}(h_t - E_{t-1}h_t).$$

$$\Delta c_t = \frac{r}{1+r} \sum_{s=0}^{\infty} \underbrace{[E_t y_{t+s} - E_{t-1} y_{t+s}]}_{News}.$$

So we have that $\Delta c_t = \frac{r}{1+r} *$ (News about human wealth at time $t$).

## 6.2 Special Case: $AR(1)$ income

- Let $y_t$ be a first order autoregressive process ($AR(1)$).

$$y_t = \alpha y_{t-1} + \epsilon_t, \quad \alpha \in [0, 1], \quad \epsilon \sim iid(0, \sigma_\epsilon^2).$$

- If $\alpha = 0$, $y_t = \epsilon_t \Longrightarrow$ White Noise.
- If $\alpha = 1$, $y_t = y_{t-1} + \epsilon_t \Longrightarrow$ Random Walk.
- Take expectations:

$$E_t[y_{t+1}] = E_t[\alpha y_t + \epsilon_{t+1}].$$

$$E_t[y_{t+1}] = \alpha y_t.$$

$$E_t[y_{t+2}] = E_t[\alpha y_{t+1} + \epsilon_{t+2}].$$

$$E_t[y_{t+2}] = \alpha E_t[y_{t+1}] = \alpha \alpha y_t = \alpha^2 y_t.$$

  In general:

$$E_t[y_{t+k}] = \alpha^k y_t.$$

- So what is the persistence of shocks? Consider:

$$\frac{\partial E_t[y_{t+k}]}{\partial \epsilon_t} = \frac{\partial E_t[y_{t+k}]}{\partial y_t} * \frac{\partial y_t}{\partial \epsilon_t} = \alpha^k * 1 = \alpha^k.$$

  So for a white noise process, $\alpha = 0$, and all shocks are purely transitory (they do not effect of beliefs about future labor income at any horizon). And for a random walk process, $\alpha = 1$, all shocks are permanent and $E_t[y_{t+k}] = y_t \ \forall \ k$. Finally, for $\alpha \in (0, 1)$, shocks are persistent but NOT permanent. See G-6.1.

- Given:

$$h_t = E_t \sum_{s=0}^{\infty} (1+r)^{-s} y_{t+s}.$$

$$h_t = \sum_{s=0}^{\infty} (1+r)^{-s} E_t y_{t+s}.$$

  We have:

$$h_t = \sum_{s=0}^{\infty} (1+r)^{-s} \alpha^s y_t.$$

$$h_t = y_t \sum_{s=0}^{\infty} \left[ \frac{\alpha}{(1+r)} \right]^s.$$

  By the infinite sum rule,

$$h_t = y_t \frac{1}{1 - (\alpha/(1+r))}.$$

$$h_t = y_t \frac{1+r}{1+r-\alpha}.$$

  Therefore, since:

$$c_t = \frac{r}{1+r}(a_t + h_t),$$

$$c_t = \frac{r}{1+r} a_t + y_t \frac{r}{1+r} * \frac{1+r}{1+r-\alpha}.$$

$$c_t = \frac{r}{1+r}a_t + \frac{r}{1+r-\alpha}y_t.$$

And since:

$$\Delta c_t = \frac{r}{1+r}(h_t - E_{t-1}h_t).$$

$$\Delta c_t = \frac{r}{1+r}\left(y_t\frac{1+r}{1+r-\alpha} - E_{t-1}y_t\frac{1+r}{1+r-\alpha}\right).$$

$$\Delta c_t = \frac{r}{1+r-\alpha}\left(y_t - E_{t-1}y_t\right).$$

$$\Delta c_t = \frac{r}{1+r-\alpha}\left((\alpha y_{t-1} + \epsilon_t) - (\alpha y_{t-1})\right).$$

$$\Delta c_t = \frac{r}{1+r-\alpha}\epsilon_t.$$

## 6.3 Empirical Testing

- What is the relationship between consumption changes and income changes?
  - Under Certainty in the LCH: $\Delta c \perp \Delta y$, or the change should be uncorrelated.
  - Under Uncertainty, we can split a time-series random variable into 2 components:

    $$x_t = \underbrace{E_{t-1}x_t}_{Forecastable} + \overbrace{x_t - E_{t-1}x_t}^{Innovation}.$$

    Thus for $\Delta y_t$:

    $$\Delta y_t = E_{t-1}[\Delta y_t] + [\Delta y_t - E_{t-1}\Delta y_t].$$

    $$\Delta y_t = E_{t-1}[y_t - y_{t-1}] + (y_t - y_{t-1} - E_{t-1}[y_t - y_{t-1}]).$$

    $$\Delta y_t = E_{t-1}[y_t - y_{t-1}] + (y_t - y_{t-1} - E_{t-1}[y_t] + y_{t-1}).$$

    $$\Delta y_t = E_{t-1}[\Delta y_t] + \underbrace{(y_t - E_{t-1}[y_t])}_{Innovation\ to\ LEVEL}.$$

    So the innovation in the change of income is equal to the innovation in the level of income. And this makes sense because our only "news" should be at time $t$ since the income at time $t-1$ should be known.

- So in the AR(1) case:

$$\Delta y_t = E_{t-1}[y_t - y_{t-1}] + (\alpha y_{t-1} + \epsilon_t - E_{t-1}[\alpha y_{t-1} + \epsilon_t]).$$

$$\Delta y_t = E_{t-1}[y_t] - y_{t-1} + \underbrace{\epsilon_t}_{y_t - E_{t-1}y_t:\ Innovation\ in\ y}.$$

Take expectations:

$$E_{t-1}\Delta y_t = E_{t-1}[y_t] - y_{t-1}.$$

$$E_{t-1}\Delta y_t = \alpha y_{t-1} - y_{t-1}.$$

$$E_{t-1}\Delta y_t = -(1-\alpha)y_{t-1}.$$

Thus, for $\alpha < 1$, we expect income to fall if current income is high and expected income should rise if current income is low. Hence income is MEAN-REVERTING.

- Also, if $\alpha = 1$, (Random Walk)

$$E_{t-1}\Delta y_t = 0 \Longrightarrow \text{ No Mean Reversion.}$$

- And, if $\alpha = 0$, (White Noise)

$$E_{t-1}\Delta y_t = -y_{t-1} \Longrightarrow \text{ Perfect Mean Reversion.}$$

- So for this case of quadratic utility, $T = \infty$, $(1+r)\beta = 1$, $y$ as an $AR(1)$ process, we have the following:
  - (1) Consumption changes are orthogonal to the predictable part of income changes.
  - (2) Consumption changes should be positively related to the innovation in income and this relationship is stronger, the higher is $\alpha$. Thus:
    * $\alpha$ high $\Rightarrow$ Income Shocks Persistent and Consumption Reponse is Large.
    * $\alpha$ low $\Rightarrow$ Income Shocks Transitory and Consumption Reponse is Small.
    * $\alpha = 1$, Random Walk,

$$\Delta c_t = \frac{r}{1+r-\alpha}\epsilon_t = \epsilon_t.$$

    * $\alpha = 0$, White Noise,

$$\Delta c_t = \frac{r}{1+r-\alpha}\epsilon_t = \frac{r}{1+r}\epsilon_t.$$

## 6.4 Bellman's Equation

- Consider the following general Bellman's Equation:

$$V_t(x_t) = max_{u_t}\Big\{r_t(x_t, u_t) + \beta E_t V_{t+1}(x_{t+1})\}, $$

subject to:

$$x_{t+1} = g(x_t, u_t, \epsilon_{t+1}).$$

Where

  - $x_t$ is a vector of state variables, eg $(a_t, y_t, r_{t+1})$,
  - $u_t$ is a vector of control variables, eg $(c_t, \phi_t)$,

- $r_t(\cdot)$ is the return function, eg $u(c)$,
- $g(x_t, u_t, \epsilon_{t+1})$ is the State Transition Equation which may depend on a vector of shocks, $\epsilon_{t+1}$: eg DBC: $a_{t+1} = R_{t+1}(a_t+y_t-c_t)$, or $AR(1)$ income: $y_{t+1} = \alpha y_t+\epsilon_{t+1}$.

- The goal is to solve for a policy function (decision rule): $u_t(x_t)$. We can usually use the Euler Equation to *characterize* the solution $u_t(x_t)$ but ideally we would like to go beyond eulers and solve for the decision rules themselves.
  - Analytic Examples: Quadratic utility (Combine Eulers with IBC), No Risky Income Model (Solve using Guess and Verify).
  - Numerical Examples: Solve these using repeated iteration.
- With a finite horizon, we know $V_T(x_T) = u(a_T + y_T)$. Plug this into our Bellman's Equation for $V_{T-1}(x_{T-1})$ and solve. Repeat.
- With an infinite horizon, suppose that the uncertainty is MARKOV. The value function and policy functions will be time invariant functions of a small number of state variables (eg $V_t(x_t) = V(a_t, y_t)$). Provided $r(\cdot)$ and $g(\cdot)$ satisfy certain regularity conditions (Blackwells) we can find ($V(x)$, $u(x)$) by repeated iteration on the Bellman's equation outlined here:
  - Step 1: Choose some bounded continuous $V^0(x)$, eg $V^0(a, y) = u(a + y)$.
  - Step 2: Iterate on:
  $$V^{j+1}(x) = max_u\{r(x, u) + \beta EV^j(g(x, u, \epsilon))\}.$$
  - Step 3: In the limit (under the contraction mapping theorem), this process will converge to:
  $$V^{t+1}(x) = V^t(x) = V^\infty = V(x).$$

## 6.5 Example: L+S Chapter 3

- Consider the following problem: $T = \infty$, $\phi_t = 0$, $r_t = r$. $y_t$ follows a markov chain with 2 states:
$$y^1 = 0, \quad and \quad y^2 = w.$$
And there is a $2x2$ transition matrix, $P$. We also have the dynamic budget constraint:
$$a_{t+1} = (1 + r)a_t + y_t - c_t.$$
Note that this DBC only multiplies $a_t$ by $1 + r$ and this just means that asset income is received during the period instead of all at the beginning.
- Notice there are only two states: $a_t$ and $y_t$ because with an infinite horizon, age $(T-t)$ is not a state, since there are no risky investments, $\phi_t = 0$ so $z_{t+1}$ is not a state. And finally since $y$ is markov, $y_t$ is a sufficient statistic for future probability distributions $\pi(y_{t+s})$ $\forall$ $s$.

- Write the Bellman's Equation (Given $a_t$ and $y_t = y^i$) as:

$$V(a_t, y^i) = max_{a_{t+1}} \left\{ u[(1+r)a_t + y^i - a_{t+1}] + \beta \sum_{j=1}^{2} P_{ij} * V(a_{t+1}, y^j) \right\}.$$

- More next time.

# 7 Lecture 7: September 22, 2004

- We continue the example started last week of solving for the value function in an infinite horizon problem via backwards iteration on the Bellman's Equation.

- Note that $T = \infty$, $\phi = 0$, and $r_t = r$. We have the first order markov process for income, $y_t : y^1 = 0, y^2 = w$, with transition matrix $P$. The Bellman's Equation (BE) given $a_t$ and $y_t = y^i$, is:

$$V(a_t, y^i) = max_{a_{t+1}} \left\{ u[\underbrace{(1+r)a_t + y^i - a_{t+1}}_{c_t}] + \beta \sum_{j=1}^{2} P_{ij} V(a_{t+1}, y^j) \right\}.$$

  Note the BE is time invariant and $P_{ij}$ is the probability of being in income state $i$ today and in state $j$ tomorrow.

- In general we solve this BE by guessing a $V^0$ and plugging it into the RHS, solving for $V^1$ on the LHS, plugging $V^1$ back into the RHS, solving for $V^2$ on the LHS, etc etc.

- How do we solve the BE in this case? Unless $u(\cdot)$ is quadratic, there is no closed form analytic solution to this BE even if we are given $V^0$.

- The solution is to "Discretize The State Space." Note this will only be an approximation. Assume $a_t \in A$ with $A = \{a_1 < a_2 < \cdots < a_n\}$. So our financial wealth only takes on a finite number of possible states. As we increase the grid (larger $n$), the approximation will get better.

- With $n$ possible asset values, there are $2 * n$ possible states (because of the 2 states of income). Thus $V(a, y)$ is an $Nx2$ matrix.

- So select particular numerical parameter values for $w$ (wage), $r$, $\beta$, $P$, and the form of the utility function and also form an initial guess at the form of $V^0$ ($Nx2$). Plug $V^0$ into the RHS and solve the BE by finding the $a_{t+1} \in A$ that maximizes the RHS of the BE for each of the $Nx2$ possible states $(a, y)$.

- This will yield a decision rule, $a^0$, also $Nx2$, giving the optimal $a_{t+1}$ for each possible state. It also yields a value function $V^1$, an $Nx2$ matrix. Plug $V^1$ into the RHS and solve for $V^2$, $a^1$, and repeat until convergence.

## 7.1 Empirical Work - Hall (1978)

- Hall tried to test the LCH using aggregate data. He assumes quadratic utility, $\beta(1 + r) = 1$, the LCH is true and there are no liquidity constraints.

- Recall the Euler equation:

$$c_t = E_t c_{t+1}.$$

  And rewrite this as:

$$c_{t+1} = c_t + \epsilon_{t+1}, \quad E_t[\epsilon_{t+1}] = 0.$$

We call consumption a Martingale under this definition. If $\epsilon_{t+1}$ is *iid* white noise, then $c$ is a random walk.

- To test, run the following OLS regression:

$$c_{t+1} = \alpha_0 + \alpha_1 c_t + \beta z_t + \epsilon_{t+1},$$

where $z_t$ is a vector of variables in the information set at time $t$. Under the null, $\alpha_0 = 0$, $\alpha_1 = 1$, $\beta = 0$. The last hypothesis says that no variables in the information set at time $t$ should be able to predict $c_{t+1}$ aside from $c_t$.

- Data: US National Income and Product Accounts (NIPA), quarterly 1948-1977. Hall measures consumption as real spending on services and non-durable goods. Note he omits spending on durable goods like cars and houses because durable spending will NOT obey the martingale (see homework 4). This means there is an implicit assumption that utility between durables and non-durables is SEPARABLE!

- First Estimation:

$$c_{t+1} = \underbrace{8.2}_{(8.3)} + \underbrace{1.130}_{(0.092)} c_t - \underbrace{0.040}_{(0.142)} c_{t-1} + \underbrace{0.030}_{(0.142)} c_{t-2} - \underbrace{0.113}_{(0.093)} c_{t-3}.$$

So here $z_t$ is lags of consumption. The result is fairly good and it looks like we can't reject the null.

- Second Estimation:

$$c_{t+1} = \underbrace{-23}_{(11)} + \underbrace{1.076}_{(0.047)} c_t + \underbrace{0.049}_{(0.043)} y_t - \underbrace{0.051}_{(0.052)} y_{t-1} - \underbrace{0.023}_{(0.051)} y_{t-2} - \underbrace{0.024}_{(0.037)} y_{t-3}.$$

Note here we do NOT include $y_{t+1}$, current income, because it is not in the information set at time $t$. In the LCH, $c_{t+1}$ will be correlated with the INNOVATION to $y_{t+1}$. The result: we reject only $\alpha_0 = 0$ but otherwise ok.

- Third Estimation:

$$c_{t+1} = \underbrace{-22}_{(8)} + \underbrace{1.012}_{(0.004)} c_t + \underbrace{0.223}_{(0.051)} p_t - \underbrace{0.258}_{(0.083)} p_{t-1} + \underbrace{0.167}_{(0.0083)} p_{t-2} - \underbrace{0.120}_{(0.051)} p_{t-3}.$$

Where $p_t$ is the real $S\&P$ 500. Result: reject of the null on several accounts. Lags of stock prices seem to be good predictors of consumption.

## 7.2 Critique of Hall's Paper

- Are there Type I errors (reject $H_0$ when it is true?) Note that Hall's main null hypothesis is that the LCH is true but there are auxiliary assumptions as well: quadratic utility, constant interest rate, $\beta(1+r) = 1$, and separability of utility between durables and nondurables. So could a failure of one of the auxiliary assumptions lead to a rejection of the main hypothesis?

- YES: if $\beta(1+r) \neq 1$. Then $\alpha_0 = 0$ and $\alpha_1 = 1$ need not be true. You will get a drift in consumption where if $\beta(1+r) > 1$, consumption should optimally rise over time. So we would say that these two assumptions on $\alpha_0$ and $\alpha_1$ are NOT robust to small variations in this auxiliary assumption.
- However, $\beta(1+r) \neq 1$ would not generate a significant coefficient on $z_t$ so the assumption that $\beta = 0$ is more robust to this auxiliary assumption.
- Now suppose that $r_t$ is time variant. Under the LCH, consumption growth should depend positively on the interest rate. Thus if $z_t$ is correlated with $r_{t+1}$ then we could estimate significant coefficients on the $z$'s even if the LCH were true. This is a result of Omitted Variable Bias and the solution is simply to include the interest rate on the right hand side since $r_{t+1}$ is in the information set at time $t$. (However, it turns out that including $r_{t+1}$ in regression (3) does not effect the significance of the stock price variables.)

- What about Type II errors (fail to reject $H_0$ when it is false)? We say that a regression has HIGH POWER if it has a LOW probability of a type II error. In principal, any vector of $z$'s in the information set could be included in Hall's regression (lags of $c$, lags of $y$, lags of durable consumption, etc etc). But what makes a set of $z$'s better than another set? Under just the null, all sets are equally valid. Under a POWER argument however, we would like to have the power to reject the null under plausible alternatives.

  - Consider the mypoia/Keynesian behavior or Liquidity constraints. Here $\Delta c_{t+1}$ might be correlated with the predictive component of $\Delta y_{t+1}$. So from a POWER standpoint, the optimal $z$'s should be variables that have substantial predictive power for future income growth.
  - In fact, we know that lags of $c$ and lags of $y$ are lousy predictors of $\Delta y_{t+1}$. In a univariate regression, aggregate output looks a lot like a random walk. However, stock prices, $p_t$, are EXCELLENT predictors of income growth. Thus Hall's third regression is much more powerful than his first two. In fact, since regression (3) rejected the LCH, under a POWER argument, Hall's paper should really say that the LCH model is a poor model of consumption.

# 8 Lecture 8: September 27, 2004

## 8.1 Review of Hall

- Hall (1978) used aggregate data to test the LCH. He used the model:

  $$c_{t+1} = \alpha_0 + \alpha_1 c_t + \beta z_t + \epsilon_{t+1},$$

  And his null hypothesis was: LCH true, quadratic utilty, $\beta(1+r) = 1$, durable/nondurable consumption separable in the utility function. Under the null:

  $$\alpha_0 = 0, \quad \alpha_1 = 1, \quad \beta = 0.$$

  Note the first two restrictions are NOT robust to type I errors since relaxing say the constant interest rate will lead to these not necessarily holding. The third restriction is more robust.

- Hull's main finding is that you can't reject $\beta = 0$ for $z_t$ = lags of consumption or $z_t$ = lags of income, but you can reject for $z_t$ = lags of stock prices.

- Interpretation: Power is a major consideration here (Pr(Type II Error)). Even if the LCH is false, we will find that $\beta \neq 0$ only if $z_t$ is correlated with $\Delta y_{t+1}$. This is because it is likely that consumption tracks changes in income so whatever we put in the $z$'s must be correlated with $\Delta y_{t+1}$ to pick up the correlation. Hence the $z$'s need to be good forecasting variables and it turns out that lagged consumption and lagged income are very poor forecasters ($\Delta y_{t+1}$ actually turns out to be *almost* a random walk so it holds no predictive power).

## 8.2 Wilcox Paper

- Wilcox also used aggregate data but he uses a very clever $z_t$. Since 1965, all changes in the social security benefit have been announced at least a few months in advance. Let $t_0$ be the announcement date and $t_1$ be the effective dates of these benefit changes. For non-durable consumption, consumption may jump at $t_0$, but not at $t_1$ because changes to consumption should be incorporated immediately at time $t_0$. For Durable consumption, consumption my also jump at $t_0$, but if anything, it should then fall at $t_1$ because usually durable purchases have some lasting (durable) quality about them so once you have them, you don't need to continue buying. Faced with an increase in income, consumers should increase consumption of durables at $t_0$ but then have nothing left to buy so $t_1$ durable consumption should actually fall.

- Data: 1963-1983 monthly aggregate real US retail sales.

- Figure 2 of the handout shows the change in the log of durable sales over time and there does appear to be some positive bias for durable spending at time $t_1$. This would go against the LCH. Table 2 presents the regression coefficients and it shows that the RHS variable, Old Age and Survivor's Income (OASI) is significant which means that the LCH for durables is rejected. Why might this be? Liquidity constraints. People cannot borrow against their future SS income so they can't buy that car at $t_0$ but instead must wait until $t_1$.

## 8.3 Aggregation

- Both Hall and Wilcox use aggregate data and both reject the LCH. But what about at the individual household level? When will rejecting the LCH at the macro level imply that it also doesn't hold at the micro level? There are two special cases: Quadratic Utility and Complete Markets Endowment Economy.

### Quadratic Utility

- Suppose $i = 1 \dots I$ consumers. Then:

$$c_{it} = E_{it}[c_{i,t+1}] = E_t[c_{i,t+1}] \;\forall\; i.$$

Note that we can remove the $i$ from the expectation if we assume "common information" across consumers. This implies:

$$c_t = \sum_{i=1}^{I} c_{it} = E_t \sum_{i=1}^{I} c_{i,t+1} = E_t[c_{t+1}].$$

So Hall's result of rejecting the LCH at the aggregate level suggests it would also be rejected at the household level.

### Complete Markets Endowment Economy (LS Ch 7)

- Suppose $i = 1 \dots I$ consumers. Let $s_t$ be a vector of random variables realized at time $t$. Let $s^t$ denote the histories of these $s$'s from $t = 0 \dots t$. Assume $\pi(s^t)$ is the probability a particular history will occur.

- Assume each household received an endowment:

$$y_{it} = y^i(s^t).$$

- So for example, with two households, 2 endowments: $y_t^1, y_t^2$ which are iid RV's which equal $H$ with probability $p$ and $L$ with probability $(1-p)$. Therefore $s_t$ is the set containing the realizations of these endowments and will have 4 realizations: $HH, HL, LH, LL$. $s^t$, on the other hand, will have $4^t$ realizations for a given time horizon, $t$.

- So make the following assumption regarding asset markets: At time 0, households can buy or sell a complete set of state-contingent **Arrow-Debreu Securities**. Security $s^t$ is a promise to pay one unit of output at date $t$ if history $s^t$ has been realized and pays zero otherwise. Assume that the securities are **Zero Net Supply** meaning that for every holder of security $s^t$, these exists an issuer to match. In other words, "for every claim, there is an obligation."

- Let the $t = 0$ price of security $s^t$ be denoted:

$$q^0(s^t).$$

In equilibrium, these prices will adjust to maintain market clearing of each security such that:

$$\sum_{i=1}^{I} c^i(s^t) = \sum_{i=1}^{I} y^i(s^t) \;\; for\ any\ s^t.$$

The implicit assumption here is that the endowment is perishable so no aggregate saving is possible.

- Household's Problem: At time 0, household $i$ takes $q^0(s^t)$ as given and chooses a state-contingent consumption plan to:

$$max \sum_{t=0}^{\infty} \sum_{s^t} \beta^t u(c^i(s^t))\pi(s^t),$$

subject to:

$$\sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t) c^i(s^t) = \sum_{t=0}^{\infty} \sum_{s^t} q^0(s^t) y^i(s^t).$$

The LHS of the constraint is the total spending to finance my state contingent consumption plan. The RHS is the total revenue raised at date 0 by pledging your state-contingent output via issuing securities.

- The beauty part about this maximization is that it is a static choice problem. All trading occurs at date 0. Thus we can solve using lagrangians.

- Let $\mu^i$ be household $i$'s lagrange multiplier. The FOC for $c^i(s^t)$ is:

$$\beta^t u'(c^i(s^t))\pi(s^t) = \mu^i q^0(s^t) \;\; \forall\, i, t, s^t.$$

The LHS is the expected discounted marginal benefit from purchasing one more $s^t$ security at date 0 and the RHS is the marginal cost of purchasing one more $s^t$ security (note that $\mu^i$ is the utility value of buying one more unit).

- Since this FOC is true for all consumers, consider the ratio of consumer $i$'s FOC to consumer $j$'s:

$$\frac{u'(c^i(s^t))}{u'(c^j(s^t))} = \frac{\mu^i}{\mu^j} \; \forall\, s^t.$$

So the ratio of marginal utilities is equal to the ratio of lagrange multipliers, but note it does not depend on the state.

- Exercise. Consider CRRA utility:

$$u(c) = \frac{\sigma}{\sigma - 1} c^{(\sigma-1)/\sigma}.$$

Thus,

$$u'(c) = c^{-1/\sigma}.$$

Thus, the ratio of FOC's becomes:

$$\frac{c_i^{-1/\sigma}}{c_j^{-1/\sigma}} = \frac{\mu^i}{\mu^j}.$$

Define the coefficient (constant) of relative risk aversion as $\gamma = \frac{1}{\sigma}$. Thus,

$$\frac{c_i}{c_j} = \left(\frac{\mu^i}{\mu^j}\right)^{-1/\gamma}.$$

$$c_i = c_j * \left(\frac{\mu^i}{\mu^j}\right)^{-1/\gamma}.$$

$$c^i(s^t) = c^j(s^t)K^{ij}.$$

So if individual $i$ is wealthier than individual $j$, then consumption for $i$ will just be a scaled up version of the consumption for $j$. Note the constant, $K^{ij}$, depends on the endowments of consumers $i$ and $j$, but does not depend on time or the current state. Thus, write:

$$c^i(s^t) = K^a c(s^t),$$

where,

$$c(s^t) = \sum_{i=1}^{I} c^i(s^t) = \sum_{i=1}^{I} y^i(s^t) = y(s^t).$$

This is true for CRRA utility and a broad class of others.

- Main Results:
  - All $c_t^i$ are perfectly correlated with aggregate consumption.
  - Households have full insurance against all idiosyncratic shocks: $c_t(s^t)$ depends on $y(s^t)$ but NOT on $y^i(s^t)$. In good idiosyncratic states with high $y^i(s^t)$, I will plan to sell more of my endowment than I consume so I become a **net issuer of securities**. In bad idiosyncratic states with low $y^i(s^t)$, I will become a **net purchaser of securities**.

# 9 Lecture 9: September 29, 2004

## 9.1 Review of Complete Markets Endowment Economy

- Under complete markets, there is a full set of $s^t$ state contingent securities traded at date 0. This allows for any possible contingency. The result is that under complete markets, individual consumption, $c^i$, tracks aggregate consumption, $c$, perfectly, which means we can use aggregate consumption to test the LCH.

- Consider one extension of the model involving Taste Shocks. Assume the original utility of the form:
$$u[c^i(s^t)v^i(s^t)].$$
Where $v^i(s^t)$ captures taste shifts for the individual or household such as health or family size. In this case the euler equation will have the following form:
$$v^i(s^t)u'[c^i(s^t)v^i(s^t)] = K^{ij}v^j(s^t)u'[c^j(s^t)v^j(s^t)].$$
So the marginal utility that is smoothed across $s^t$ depends also on taste shifts.

- The main results of the Complete Markets analysis are as follows:
  - (1) The individual's consumption tracks aggregate consumption perfectly.
  - (2) The individual's consumption is fully insured against all idiosyncratic risk to $y^i(s^t)$. Note this is a stronger statement than the typical LCH statement that individual consumption is orthogonal to predictable changes in individual income. However, still under complete markets, $\Delta c^i$ will be correlated with innovations to $\Delta y^i$, but only through the non-idiosyncratic part. $\Delta c^i$ should be uncorrelated with the idiosyncratic component of innovations to $\Delta y^i$.

- There are 2 caveats to all of this though:
  - (1) $\Delta c^i$ will be correlated with aggregate $\Delta y$ even under complete markets (so there is not complete insurance against everything ... if there is a bad state for everyone, no one can sell securities to insure themselves against this bad state since there won't be any buyers under the no-excess supply assumption).
  - (2) $\Delta c^i$ may be correlated with the idiosyncratic component to innovations in $\Delta y^i$ ONLY IF $\Delta y^i$ is correlated with taste shocks, $v^i(s^t)$.

- So under complete markets, $c^i$ will respond to idiosyncratic taste shocks $v^i$, (eg, family size will effect optimal $c^i$ and may also effect $y^i$).

## 9.2 Testing the Complete Markets Economy

- Why might the complete markets hypothesis fail?
  - (1) Hidden Information: $s^t$ may not be fully and symmetrically observed by everyone. Suppose $s_t$ represented individual income of $N$ households at time $t$. If this

was private information to household $i$, then we won't be able to trade securities contingent on these individual incomes because people will have an incentive to lie and claim low income (so they don't have to pay off their securities).

  - (2) Limited Enforcement: $s^t$ is fully observed but contracts are not automatically enforced.

**Cochrane Paper**

- Cochrane asks: Are households fully insured against idiosyncratic risk?

- Data: Panel Study of Income Dynamics (PSID). Tracks fixed group of households and their off-spring since 1968. It's the best data to test intergenerational linkages. Has rich information on labor market behavior, but consumption data is limited and noisy. There is also occasional wealth data.

- Sample: 4629 households observed from 1980 to 1983. And a subsample of 1741 households which have no composition changes in the same period (no marriages, divorces, births, deaths, etc).

- OLS regression:
$$\Delta c_i = \alpha + \beta x_i + \epsilon_i.$$
Where $\Delta c_i$ is the percentage change in household $i$'s food consumption between 1980 and 1983, $\alpha$ is a common coefficient which reflects aggregate shocks and interest rates. And $x_i$'s are measures of idiosyncratic household level shocks, (eg unemployment). Hence under the null hypothesis of complete markets being true:
$$\beta = 0.$$

- Econometric issues:
  - (1) Consumption data is noisy. Since consumption is a LHS variable, this will NOT bias $\hat{\beta}$ but will cause $se(\hat{\beta})$ to be higher than normal. This will increase the chances of failing to reject $\beta = 0$ when the null is false (P(type II error)).
  - (2) Taste shocks. Even under the null, $\beta$ may not be 0 for $x_i$'s that are correlated with taste shocks (eg, family size). So this increases the chance of rejecting $\beta = 0$ even when the null is true (P(type I error)).

- The solution to this second point might be to either use households without composition changes (which should reduce taste changes) or to possibly use $x_i$'s which are plausibly uncorrelated with taste shocks.

- The results shown in the handout are as follows. Regression 6 and 7 use the changes in family size and then income growth as the $x_i$'s. Cochrane includes these not because he thinks that $\beta$ will actually be zero, but rather because he knows it shouldn't be and this will convince the reader that the problem of noisy consumption data is not that bad. Indeed, in both these regressions, $\beta$ is significantly different from zero, which could still be consistent with Complete Markets and it has eased our concern about econometric issue number 1 above.

- In the other regressions, we reject $\beta = 0$ for $x_i$'s equal to involuntary job loss and illness, but for some reason we cannot reject when using strike days.
- Overall, complete markets is rejected and this implies that household level consumption does not aggregate up, so this suggests we need to test the LCH model in household data.

## 9.3 Micro-level Tests of the LCH

**Zeldes (1989 JPE)**

- Zeldes tests for liquidity constaints (LC) in the PSID. Liquidity constraints prevent borrowing but should not prevent dissaving. So if LCs are present, the LCH euler equation may fail for households without liquid wealth, but should be satisfied for households with liquid wealth. The euler equation will fail only for households facing a binding LC.
- OLS regression:

$$\Delta log(c_{it}) = \alpha_{0i} + \alpha_1 (AGE)_{i,t-1} + \alpha_2 (Annual\ Food\ Needs)_{it} + \frac{1}{\rho} log(1+r_{it}) + \lambda y_{i,t-1} + \epsilon_{it}.$$

- Note, Cochrane used just two points in time for his regression, 1980 and 1983, where Zeldes examines these families over time so the intercept coefficient is household specific, and should relfect variations in the time discount rates. Note that the interest rate is also household specific because $(1+r_{it})$ is now the measure of after tax expected real US treasury bill returns using household $i$'s tax rate.
- Data: 12-13,000 observations on 4,000 PSID households, pooled (1968-1982) annual data.
- Zeldes estimates for two samples:
  - Poor Sample: Households with less than 2-months of income in wealth holdings.
  - Rich Sample: Households with more than 2-months of income in wealth holdings.
  - Note that he further delineates between liquid wealth including housing and liquid wealth excluding housing.
- Null: $\lambda = 0$. Under the LCH, in both the rich and poor samples, since $y_{i,t-1}$ is in the information set at time $t$, there should be no impact on changes in consumption. Alternatives: Under Keynesian myopia, $\lambda$ may be non-zero for either or both groups. Under liquidity constraints, $\lambda = 0$ for rich households and $\lambda \neq 0$ for poor.

- Results: Table 2 reports the results where sample is split according to non-housing wealth.

$$\lambda < 0 \text{ Significant, for Poor Sample.}$$

$$\lambda < 0 \text{ Insignificant, for Rich Sample.}$$

- Results: Table 3 reports the results where sample is split according to total wealth (including housing).

$$\lambda < 0 \text{ Significant, for Poor Sample.}$$

$$\lambda < 0 \text{ Significant, for Rich Sample.}$$

- So using non-housing wealth, we get a rejection of the LCH model only for the poor sample (which correponds to the liquidity constraint argument on poor households). Using total wealth, we reject LCH outright.

# 10 Lecture 10: October 4, 2004

## 10.1 More on Zeldes

- Recall the regression in Zeldes (1989):

$$\Delta log(c_{it}) = \alpha_{0i} + \alpha_1 (AGE)_{i,t-1} + \alpha_2 (Annual\ Food\ Needs)_{it} + \frac{1}{\rho} log(1 + r_{it}) + \lambda y_{i,t-1} + \epsilon_{it}.$$

- Zeldes is trying to test the LCH against a particular alternative hypothesis, liquidity constraints. See lecture 9 for details, but we find that for liquid, non-housing, wealth, the LCH hypothesis is rejected for the poor and upheld for the rich. For a total wealth split, it is rejected for both groups. Zeldes interpretation is that there is a type I error in the second split. In reality, about 1000 households in his sample would have been poor in the first split, but because they have housing wealth (and ONLY housing wealth) they move into the rich group and bias the coefficient on $\lambda$ away from 0. The first split of just liquid wealth seems more appropriate and seems to validate the LCH for unconstrainted households.

- Shea's Critique: Do Zeldes' tests really distinguish liquidity constraints from myopia? Possibly NO due to heterogeneity in household's ability to plan ahead. If you consider that some households are more into planning, irrespective of wealth, they would be more likely to save for the future and just because they are foresighted, they would be more likely to satisfy the LCH. The other myopic households would have the opposite result – no savings and more likely to violate the LCH. So even without liquidity constraints, this could explain some of Zeldes' findings.

- Amriks, Caplin and Leahy (QJE 2003) find that myopic people save less by asking them if they plan vacations ahead of time or just figure it out along the way.

- Hurst - Ants and Grasshoppers, finds that households with little wealth at retirement also violate the LCH throughout their lives.

## 10.2 Shea (1995)

- Shea tries to find a good measure of $E_{t-1}\Delta y_t$. The idea is to match PSID households to particular long term union contracts and then use public information on union contract provisions along with inflation forecasts to measure a variable called EDWAGE : Households expected percentage change in real wages. Since union contracts often run for several years, we have a sample of households where their expected wage is determined (aside from inflation uncertainty).

- The difficult part of this was the matching. We know if a household member is a union member from the PSID and we also know their industry, occupation and location (but not the actual firm they work for). So we exploit this information using two subsamples:

  - Pattern Bargaining. Industries were all firms are part of the same union (Teamsters Trucks, Lumber industry).

  - Dominant Local Employer. If you work in the auto business in Flint Michigan, we are fairly certain that you work for GM.

- So using these households where we can identify the union they are in and their consumption and EDWAGE, we end up with 647 observations on 285 households. This is the biggest weakness of the paper (Or the "Body buried in the paper.")

- Estimate the following equation:

$$\Delta log(c_{it}) = \alpha_t + \gamma \Delta log(Food\ Needs)_{i,t} + \frac{1}{\rho} log(1 + r_{it}) + \beta z_{i,t-1} + \epsilon_{it}.$$

  Note the constant term is the same across households just due to the limited amount of data. The $z$'s are either $EDWAGE$ in table 3 - column 1, or lags of consumption or income in columns 2 thru 4. When $z = EDWAGE$, we get a positive and significant (10%) coefficient for $\beta$. For other $z$'s, we do not get significance. This suggests that the Hall/Zeldes $z$'s have relatively little power compared to $EDWAGE$. This is because $EDWAGE$ is more strongly indicative of expected income growth, $E_{t-1}y_t$.

- Table 4 attempts to determine if we have a Liquidity Constraints story or a Myopia story. In columns 1 thru 4, Shea redoes the analysis of Zeldes and splits the sample between the rich and poor households. He finds little difference between $\hat{\beta}$ for the two groups. In column 5, he allows $\beta$ to differ depending on if $EDWAGE$ was negative or positive. In other words, the split is now for households which expect to have their (REAL) income grow soon and those that expect to see a fall in (REAL) wages in the future. This is because, under liquidity constraints, the euler equation for the LCH between period $t$ and $t+1$ should only fail for households for which their optimal unconstrained consumption is greater then their income assets. These households face a binding LC. This condition is more likely to hold if

$$y_t < E_{t-1}y_t \Longrightarrow EDWAGE > 0.$$

  For households that expect their income to RISE, the LCH euler says they should consume relatively more today and smooth out consumption over their lifetime. This may not be possible with LCs. The opposite though, should not be true:

$$y_t > E_{t-1}y_t \Longrightarrow EDWAGE < 0.$$

  This should not lead to a binding LC situation because LC's do not prevent savings, just borrowing. So under LC, we should see:

$$\beta = 0 \ \ if\ EDWAGE < 0.$$

$$\beta > 0 \ \ if\ EDWAGE > 0.$$

  Under myopia, we should see $\beta > 0$ for both groups and under the LCH, $\beta = 0$, for both.

- Findings.

$$\hat{\beta} > 0 \;\; if \; EDWAGE < 0.$$
$$\hat{\beta} = 0 \;\; if \; EDWAGE > 0.$$

This is a failure of the LCH, myopia and the liquidity constraints arguments!! However, we do see the same result in aggregate data (JMCB 1995): aggregate consumption seems to be more sensitive to predictable falls in aggregate income than to predictible rises in aggregate income. One explanation might be Loss Aversion, discussed next.

## 10.3 Loss Aversion

- Consider two choices.
  - 1) Choose lottery A: Gain \$1,000 with certainty or lottery B: Gain \$100,000 with probability 0.01 and gain \$0 with probability 0.99.
  - 2) Choose lottery A: Lose \$1,000 with certainty or lottery B: Lose \$100,000 with probability 0.01 and lose \$0 with probability 0.99.

  Most people would choose (A) in the first situation (Risk aversion) and (B) in the second situation (Risk Aversion).

- This could be the result of Prospect Theory (Kahneman and Tversky 1979) and is displayed in G-10.1. Consider a utility function of the following form:

$$u(c) = V(r) + W(c - r).$$

  Where $V(\cdot)$ is your regular concave utility function over your reference (or usual) consumption, $r$. $W(\cdot)$ is concave in gains and convex in losses and it is a function of your relative gains or losses compared to your reference consumption.

- Example: Bowman, Minehart and Rabin examine a 2-period consumption model. with loss aversion and uncertain income, $y$. Let $y_1 = r$ and $y_2 = r$ with probability 0.5 and $y_2 < r$ with probability 0.5. Under standard preferences, the optimal choice would be to set $c_1 < y_1 = r$ in order to smooth a potentially large loss of income in period 2. This is the "Save It For A Rainy Day" strategy. However, under loss aversion, the consumer should set $c_1 = y_1 = r$, despite the potential income loss in period 2. This is because if income does fall, you'll have a bigger drop in consumption then (but you're risk loving in these losses) and if income doesn't fall, you keep $c = r$ in both periods and avoid the relatively large loss in utility from setting $c_1 < r$. See G-10.2.

- Loss Aversion may explain environmental policy in that we should cut carbon emissions today but since there is a chance (no matter how small) that the scientists are wrong, we would rather not have to suck it up today and face the consequences tomorrow (with the implicit hope that we were right all along). It is also the case with criminal defence strategies. Defendants and their lawyers will often reject a plee that would result in a short prison term in favor of going to trial that has a large chance of failing and going to prison for even longer, but at least having a small chance of acquital.

# 11 Lecture 11: October 6, 2004

## 11.1 Follow-up on Empirical Testing

- First of all, not all of the empirical papers completely reject the LCH under uncertainty. One possible explanation often put forward is called "Bounded Rationality": This means it is optimal to smooth large predictable swings in income, but not small swings. Since there may be costs to reoptimizing, a consumer will only reoptimize if the utility gains outweight the reoptimization costs.

## 11.2 Returning to the Theory

- Recall our baseline case of the LCH with quadratic utility, no risky assets, $T = \infty$, $\beta(1+r) = 1$, perfect capital markets, and no non-negatively constraints on consumption. Our decision rule was:

$$c_t = \frac{r}{1+r}(a_t + \sum_{s=0}^{\infty}(1+r)^{-s}E_t(y_{t+s})).$$

This decision rule (or policy function) displays **Certainty Equivlance (CE)**. This means that optimal behavior in the model is identical to behavior in an otherwise identical model in which future random variables are assumed to take on their mean values with CERTAINTY. This means that uncertainty, per se, does not effect the optimal decision rules. Note this does not imply risk neutrality. In our model, utility is reduced by uncertainty although the decision rule for consumption is the same.

- A model that demonstrates CE is craziness. Consider the following illustration. Suppose we have a 2 period model with $y_0 = 100$ and in case 1: $y_1 = 1,000,000$ with certainty while in case 2: $y_1 = 10,000,000$ with probability 0.1 and $y_1 = 0$ with probability 0.9. Under CE, we should have the same consumption rule in both cases because the expected value of income in each case is the same. Clearly, this is not realistic. If I knew I had a million dollars coming with certainty, I would be much more likely to borrow against that amount than if I had to face the lottery in case 2. No bank would even loan me the money in case 2.

- So how can we eliminate CE from the model?

**Non-Negativity Constraints**

- Suppose we impose the restriction:

$$c_t \geq 0 \; \forall \; t.$$

- Back in our example we would have $c_0, c_1 \geq 0$ which means in period 1:

$$\underbrace{(1+r)(100-c_0)}_{s_0} + \underbrace{min(y_1)}_{Worst\ Case} \geq 0.$$

Or,

$$100 - c_0 \geq -\left(\frac{min(y_1)}{1+r}\right).$$

So our initial debt can be no higher than the PDV of future income in the worst case scenario. Thus in case 1, the household can borrow up to $1,000,000/(1+r)$ and in case 2 they cannot borrow at all.

- Result: The combination of non-negativity constraints on consumption and a solvency condition $\Longrightarrow$ Endogeneous Liquidity Constraint, or sometimes called a "Natural Borrowing Constraint."

- More generally, consider a $T$ period model with no risky assets. Then non-negativity and solvency imply:

$$a_t \geq -\sum_{s=t}^{T}(1+r)^{-(s-t)} * min(y_s). \quad (*)$$

This result is regardless of which utility function we choose. Thus current debt cannot exceed your PDV of future income in the worst case scenario. So having income of zero as a possibility in all periods (no matter if the probability is small), this implies $a_t \geq 0$.

- What if we allowed for defaults in bad periods? In this case, we would have a wedge between the borrowing and lending rates of interest:

$$r_B > r_L,$$

to account for the added risk of possible default. This situation would lead to a modification of our Euler equation to:

$$\beta(1+r_L)E_t u'(c_{t+1}) \leq u'(c_t) \leq \beta(1+r_B)E_t u'(c_{t+1}).$$

- Note that CE will not hold if there are natural borrowing constraints because mean-preserving spreads to future income that reduce $min(y_{future})$ will tighten (*). Since (*) is a borrowing constraint, the usual Euler equation need not hold with equality. If (*) binds,

$$u'(c_t) \geq \beta(1+r)E_t[u'(c_{t+1})].$$

**Non-Quadratic Utility**

- The second way to deal with CE is to introduce non-quadratic utility. Suppose we have a 2 period problem with $\beta = 1 + r = 1$. Suppose utility takes the form:

$$V = u(c_0) + E_0[u(c_1)].$$

And suppose $y_0 = \mu > 0$ and $y_1 = \mu + \epsilon_1$ with $E_0[\epsilon_1] = 0$. Assume there are no non-negativity constraints on consumption. Under certainty,

$$c_0^* = c_1^* = \mu.$$

(Because $\epsilon_1 = 0$). Under uncertainty, we have:

$$u'(c_0) = E_0[u'(c_1)].$$

But $c_1 = 2\mu - c_0 + \epsilon_1$, so,

$$u'(c_0) = E_0[u'(2\mu - c_0 + \epsilon_1)].$$

Let $K_0 = 2\mu - c_0$. So,

$$u'(c_0) = E_0[u'(K_0 + \epsilon_1)].$$

Now take a $2^{nd}$ order taylor series approximation of $u'(c_1)$ around the point $c_1 = K_0$. Thus:

$$u'(c_0) = E_t[u'(K_0) + \epsilon_1 u''(K_0) + 0.5\epsilon_1^2 u'''(K_0)].$$

$$u'(c_0) = u'(K_0) + E[\epsilon_1]u''(K_0) + 0.5E[\epsilon_1^2]u'''(K_0).$$

$$u'(c_0) = u'(K_0) + 0.5\sigma_{\epsilon_1}^2 u'''(K_0).$$

- In the special case of quadratic utility, $u'''(K_0) = 0$, so:

$$u'(c_0) = u'(K_0).$$

$$c_0 = E_0[c_1] = \mu \Longrightarrow \text{Certainty Equivalence.}$$

- But for CARA or CRRA utility, $u'''(K_0) > 0$, so

$$u'(c_0) > u'(K_0).$$

$$c_0^* < E_0[c_1^*] = \mu.$$

And this is called **Precautionary Savings.** We consume less today they we expect to consume tomorrow. Sort of the save for a rainy day story. For utility functions with a positive $3^{rd}$ derivative, the optimal $c_t$ under uncertainty will be less than the CE consumption holding state variables constant.

- Why does $u'''(K_0)$ matter? If $u''' > 0$, then $u'$ is a convex function, so $u'(c)$ will increase rapidly as consumption approaches 0. So we will want to save more now to avoid such states in the future. We want to bound consumption away from zero. See G-11.1.

- By Jensen's Inequality, if $u''' > 0$,

$$E_t[u'(c_{t+1})] > u'(E_t[c_{t+1}]).$$

Thus our euler equation becomes:

$$u'(c_t) = \beta(1+r)E_t u'(c_{t+1}) > \beta(1+r)u'(E_t[c_{t+1}]).$$

If $\beta(1+r) = 1$,

$$u'(c_t) = E_t u'(c_{t+1}) > u'(E_t[c_{t+1}]).$$
$$u'(c_t) > u'(E_t[c_{t+1}]).$$
$$c_t < E_t[c_{t+1}].$$

Again, precautionary savings. So without quadratic utility, we do not smooth consumption over our lifetimes and precautionary savings reduces our current consumption today and increases our assets tomorrow and increases our expected consumption tomorrow.

# 12 Lecture 12: October 11, 2004

## 12.1 Precautionary Savings

- Consider the following CRRA utility:

$$u(c) = \frac{1}{1-\rho} c_t^{1-\rho}.$$

Recall the euler equation:

$$u'(c_t) = \beta(1+r_t)E_t u'(c_{t+1}).$$

Substituting:

$$c_t^{-\rho} = \beta(1+r_t)E_t[c_{t+1}^{-\rho}].$$

Rearrange:

$$E_t\left[\frac{c_{t+1}}{c_t}\right]^{-\rho} = [\beta(1+r_t)]^{-1}.$$

To linearize, we can take two approaches: assume log-normality and taylor series approximation.

**Log-Normality**

- Rewrite the LHS:

$$E_t exp[-\rho \Delta log(c_{t+1})].$$

- Assume $\Delta log(c_t)$ is a normal random variable. Then:

$$exp(-\rho \Delta log(c_t))$$

is log-normal. Note that if $X$ is log normal with $log(X) \sim N(\mu, \sigma^2)$, then $E[X] = exp(\mu + \sigma^2/2)$. So LHS becomes:

$$exp(-\rho E_t[\Delta log(c_{t+1})] + \frac{\rho^2 \sigma^2_{\Delta logc}}{2}).$$

The entire Euler is then:

$$exp(-\rho E_t[\Delta log(c_{t+1})] + \frac{\rho^2 \sigma^2_{\Delta logc}}{2}) = [\beta(1+r_t)]^{-1}.$$

Take logs of both sides and simplify:

$$-\rho E_t[\Delta log(c_{t+1})] + \frac{\rho^2 \sigma^2_{\Delta logc}}{2} = log[\beta(1+r_t)]^{-1}.$$

$$-\rho E_t[\Delta log(c_{t+1})] = -\frac{\rho^2 \sigma^2_{\Delta logc}}{2} - log(\beta) - log(1+r_t).$$

$$E_t[\Delta log(c_{t+1})] = \frac{\rho}{2}\sigma^2_{\Delta logc} + \frac{1}{\rho}log(\beta) + \frac{1}{\rho}log(1+r_t).$$

Which we can rewrite without the expectation as:

$$\Delta log(c_{t+1}) = \frac{\rho}{2}\sigma^2_{\Delta logc} + \frac{1}{\rho}log(\beta) + \frac{1}{\rho}log(1+r_t) + v_{t+1}, \quad E_t[v_{t+1}] = 0.$$

- So from the first term on the LHS, we have that the mean of consumption growth is increasing in the variance of consumption growth and in the the coefficient of risk aversion. So if we are more uncertain or less risky regarding the future, we should have a higher growth which implies we consume less today and more tomorrow. Precautionary savings.

**Taylor Series Expansion**

- Rewrite the euler again without the expectation sign as:

$$\left[\frac{c_{t+1}}{c_t}\right]^{-\rho} = [\beta(1+r_t)]^{-1}(1+e_{t+1}), \quad E_t[e_{t+1}] = 0.$$

Take logs,

$$-\rho log\frac{c_{t+1}}{c_t} = -log(\beta) - log(1+r_t) + log(1+e_{t+1}).$$

$$-\rho log(c_{t+1}) - log(c_t) = -log(\beta) - log(1+r_t) + log(1+e_{t+1}).$$

$$-\rho\Delta log(c_{t+1}) = -log(\beta) - log(1+r_t) + log(1+e_{t+1}).$$

$$\Delta log(c_{t+1}) = \frac{1}{\rho}log(\beta) + \frac{1}{\rho}log(1+r_t) - \frac{1}{\rho}log(1+e_{t+1}).$$

Now take a second order taylor series approximation of $log(1+e_{t+1})$ around $e_{t+1} = 0$.

$$log(1+e_{t+1}) = log(1) + \frac{d\ log(\cdot)}{de_{t+1}}(1+e_{t+1}-1) + \frac{1}{2}\frac{d^2\ log(\cdot)}{de^2_{t+1}}(1+e_{t+1}-1)^2.$$

$$log(1+e_{t+1}) = e_{t+1} + \frac{1}{2}(e_{t+1})^2.$$

Plug into the euler and take expectations:

$$E_t\Delta log(c_{t+1}) = \frac{1}{\rho}log(\beta) + \frac{1}{\rho}log(1+r_t) - \frac{1}{\rho}E_t[e_{t+1} + \frac{e^2_{t+1}}{2}].$$

$$E_t\Delta log(c_{t+1}) = \frac{1}{\rho}log(\beta) + \frac{1}{\rho}log(1+r_t) - \frac{\sigma^2_e}{2\rho}.$$

This is the same result as before if you note that $\sigma^2_e = var(\rho\Delta log(c_{t+1})) = \rho^2\sigma^2_{\Delta logc}$, so:

$$E_t\Delta log(c_{t+1}) = \frac{1}{\rho}log(\beta) + \frac{1}{\rho}log(1+r_t) - \frac{\rho^2\sigma^2_{\Delta logc}}{2\rho}.$$

$$E_t \Delta log(c_{t+1}) = \frac{1}{\rho} log(\beta) + \frac{1}{\rho} log(1 + r_t) - \frac{\rho \sigma^2_{\Delta logc}}{2}.$$

I get a minus sign here and he doesn't ... check this!!!!!

- In Zeldes' estimating equation (1989 JPE), he has:

$$\Delta log(c_{i,t+1}) = \frac{1}{\rho} log(\beta_i) + \frac{1}{\rho} log(1 + r_{i,t+1}) + \lambda \Delta x_{i,t+1} + \frac{\rho \sigma^2_{\Delta logc}}{2} + \gamma y_{it} + v_{i,t+1}.$$

  And he assumes $\sigma^2_{\Delta logc_{i,t+1}} = \sigma^2_i$, constant over time but may vary across households. So he absorbs the household specific constant along with possibly varying discount factors.

- Carroll (1997) is critical of this assumption. The variance of consumption growth is not likely to be time invariant. In particular it is likely to be a decreasing function of $E_t[a_{t+1} + y_{t+1}]$, cash on hand at time $t + 1$.

**Zeldes - 1989 QJE**

- Zeldes does a quantitative analysis of precautionary savings under CRRA utility. His model:

$$max\ E_0 \sum_{t=0}^{T} \beta^t \left( \frac{1}{1-\rho} \right) c_t^{1-\rho}.$$

  With initial cash on hand given: $x_0 = a_0 + y_0$, $a_{T+1} = 0$ and dynamic budget constraint (DBC):

$$x_{t+1} = a_{t+1} + y_{t+1} = (1 + r)(x_t - c_t) + y_{t+1}.$$

- So solve for optimal behavior, you need to use numerical dynamic programming (backwards iteration on the BE).

- See G-12.1 for his results in a $T = 15$ model with no discounting or interest rates and a risk aversion of 3. He shows that for certainty equivalence ($C_{CE}$), optimal consumption is just an increasing function of initial wealth with the $MPC = \frac{1}{15}$. Under uncertainty, if $y = \{0, 100, 200\}$ with $Prob = \{0.15, 0.70, 0.15\}$ versus, $y = \{0, 100, 200\}$ with $Prob = \{0.25, 0.50, 0.25\}$, he shows that in the second case with more uncertainty, households save more at all levels of wealth. In the first case of uncertainty, households also save, but not as much. Note that precautionary savings is the vertical distance between the $C_{CE}$ curve and the optimal consumption under uncertainty curves.

- The last main result to note is that the consumption function is CONCAVE. This means that the MPC is high for low levels of $x_0$ and declines to $\frac{1}{15}$ as $x_0 \to \infty$. (This does NOT generalize to CARA utility functions – see homework).

- Intuition: why is $MPC > 0$?

- (1) Lifetime wealth effect: this is present under CE and uncertainty and it generates the MPC of $\frac{1}{15}$.
- (2) Breathe easier effect: At low levels of $x_0$, the probability of being in a bad state with a high MPC is very high and extra $a_0$ or $y_0$ provides more insurance against these bad states and this weakens the precautionary savings motive.

- The last point to note is that $C_0(x_0) < x_0$ even for very small $x_0$. So there is no borrowing despite the absence of liquidity constraints. Optimally, agents will set $c < y$ in all periods because there exists some small chance that income will be zero from this point on. This is because for CRRA utility, $u'(0) = \infty$. So consumers will avoid the 0 consumption singleton at ALL costs! If 0 income is possible in the finite horizon ($T = 15$ in our case), then avoid borrowing in every period. Thus,

$$s_0 > -\sum_{t=1}^{15}(1+r)^{-t}min(y_t).$$

In Zeldes $min(y_t) = 0 \; \forall \; t$ so $s_0 > 0$.

- NOTICE: This is not the same as natural borrowing constraints but it is similar. In Zeldes' model, the LCH-euler will hold at all dates and all states yet we have no borrowing. In the natural borrowing constraint story, the euler does not hold with equality.

**Carroll Critique**

- So why is $\sigma^2_{log\Delta c_t}$ time varying under CRRA? Because the consumption function is CONCAVE! See G-12.2 which shows that depending on the level of wealth, or cash on hand, $x_t$, the consumption variation will be different. Write cash on hand as:

$$x_t = \underbrace{(1+r)(x_{t-1} - c_{t-1}) + E_{t-1}[y_t]}_{E_{t-1}[x_t]} + \underbrace{\epsilon_{y_t}}_{Innovation}.$$

So shocks to $x_t$ will have a bigger impact on $c_t$ when the baseline $x_t$ is low.

- The implication is that the euler equation typically estimated by empirical researchers (including Shea) is misspecified under CRRA utility. The typical estimation is:

$$\Delta log(c_t) = Constant + \lambda(Taste\ Shocks) + \sigma\ log(1+r_t) + \gamma z_{t-1} + \epsilon_t.$$

Under the LCH null: $\gamma = 0$.

- But Carroll would say that we should also include the time varying $\sigma^2_{\Delta log(c_t)}$. If we omit this variable, we could reject $H_0$ if $z_{t-1}$ is correlated with $\sigma^2_{\Delta log(c_t}$ (Omitted Variables Bias). This could especially be a problem if $z_{t-1}$ is correlated with $E_{t-1}[x_t]$.

- The implication for Zeldes' work is that precautionary wealth holdings should be very large for most households (See G-12.1). However, in fact, many households hold very little non-housing wealth. See table 1 of Hubbard 1995 which shows this result.

- So the question now becomes: How can we make precautionary savings model's implications for wealth holdings more consistent?

  - (1) Social Insurance (Hubbard 1995): Welfare, TANF, etc. This reduces income uncertainty and because these programs are usually means-tested, this generates savings disincentives for low-wealth households (you only qualify if your wealth is below a certain level).
  - (2) Impatience (Deaton 91, Carroll 97). If $\beta(1+r) < 1+g$ where $g$ is the trend growth rate of income. Even under CE, this would mean that optimal behavior for agents would be to borrow against future income. If our income is expected to grow rapidly in the future (beyond all discounting and interest), we definitely want to consume now, whatever it takes.

# 13 Lecture 13: October 13, 2004

## 13.1 Deaton 91

- Deaton's paper was one of those in response to Zeldes' paper which found that people should save a lot. Empirically, agents do not build up unrealistically larger buffer stocks of wealth.

- Deaton's model is similar to Zeldes but he sets $T = \infty$, imposes a (sometimes binding) liquidity constraint, $a_t \geq 0 \; \forall \; t$, $\beta = (1/1.1)$, $r = 0.05$, and he simulates income as coming from a discrete approximation to the normal distribution:

  $$y_t \sim N(100, \sigma).$$

  Critically, he assumes $y_t$ is STRICTLY positive. There is no possibility of having no income in any state.

- Figure 1 (G-13.1) shows various consumption lines for different levels of wealth depending on the variance of income and the risk premium parameter. Clearly, optimal $c(x)$ is lower for higher $\sigma$ and $\rho$ (more variation in income or more risk averse). Also, optimal consumption:

  $$c(x) = x_t \; \forall \; x_t \leq x^* \approx 100.$$

  In these cases, the liquidity constraint binds and the euler equation between $t$ and $t+1$ fails.

- The reason the LC binds here and not in Zeldes' paper is (1) Strictly positive income and (2) Impatience (agents are less willing to postpone consumption today).

- Figure 2 (see handout) shows a time series of consumption, income and asset holdings from a simulation of income with $\sigma = 10$, $\rho = 3$ and $a_0 = 0$. Once you simulate an income process, you can plug it into the DBC and the decision rule for consumption to find consumption and assets over time. 3 Results from this figure:

  - (1) Note that consumption is still smoother than income depsite the presence of liquidity constraints. Consumers do not accumulate large amounts of assets like in Zeldes.
  - (2) Assets are low, sometimes reverting to zero, which is much more realistic and comes out of the impatience built into the model.
  - (3) Consumption volatility is asymmetric. Consumption spikes down in times of low assets and low income (binding LC), while consumption never spikes up (a favorable income shock will be saved).

## 13.2 Asset Pricing

- The work-horse model of this literature is the Consumption-based Capital Asset Pricing Model (C-CAPM).

- The euler equation is similar to the one from consumption:

$$u'(c_t) = \beta E_t[R_{t+1} u'(c_{t+1})].$$

This holds for all assets and for all households. Now $R_{t+1}$ is the rate of return on an asset. Given that $p_t$ is the price of the asset, it may be:

$$\text{General:} \quad R_{t+1} = \frac{(\text{Payments from Issuer})_{t+1} + p_{t+1}}{p_t}.$$

$$\text{Stock:} \quad R_{t+1} = \frac{(\text{Dividends})_{t+1} + p_{t+1}}{p_t}.$$

$$\text{1-period discount bond:} \quad R_{t+1} = \frac{(\text{Face Value})_{t+1}}{p_t}.$$

$$\infty\text{-lived consol bond:} \quad R_{t+1} = \frac{(\text{Coupon})_{t+1} + p_{t+1}}{p_t}.$$

- Rewrite the euler as follows:

$$\frac{1}{\beta} = \frac{1}{u'(c_t)} E_t[R_{t+1} u'(c_{t+1})].$$

$$\frac{1}{\beta} = E_t[R_{t+1} \frac{u'(c_{t+1})}{u'(c_t)}].$$

$$\frac{1}{\beta} = E_t[R_{t+1} G_{t+1}].$$

Where,

$$G_{t+1} = \frac{u'(c_{t+1})}{u'(c_t)}.$$

For example, for CRRA utility, $G_{t+1} = (c_{t+1}/c_t)^{-\sigma}$.

- Now expand the expectation:

$$\frac{1}{\beta} = E_t[R_{t+1}] E_t[G_{t+1}] + cov_t(R_{t+1}, G_{t+1}).$$

Solve for $E_t R_{t+1}$:

$$E_t[R_{t+1}] = \frac{\frac{1}{\beta} - cov_t(R_{t+1}, G_{t+1})}{E_t[G_{t+1}]}.$$

And this is the key C-CAPM equation.

**Application 1: Risk Free Interest Rate**

- Suppose we are studying fluctuations in the risk free interest rate. By definition, the covariance term will be zero for risk free bonds, so:

$$E_t[\overline{R}_{t+1}] = \overline{R}_{t+1} = \frac{1/\beta}{E_t[G_{t+1}]} = \frac{1}{\beta E_t[G_{t+1}]}.$$

- So how would the riskless rate fluctuate in an endowment economy with no idiosyncratic risks?

- Assume many identical consumers with $u(c) = log(c)$ so that:

$$G_{t+1} = \left(\frac{c_{t+1}}{c_t}\right)^{-1}.$$

  Assume each consumer receives an identical perishable endowment every period:

$$y_t = \mu_t + \epsilon_t, \quad \epsilon_t \sim (0, \sigma^2).$$

  Because the endowment is perishable, no saving is possible in the aggregate so:

$$C_t = Y_t \; \forall \; t.$$

- In principal, agents are allowed to borrow and lend, (buy and sell) riskless 1 period bonds. Thus $c_t$ need not equal $y_t$ in all periods. However, in practice, in equilibrium, everyone is identical so $c_t = y_t$ for all individuals and no borrowing will occur.

- So what will be the risk free interest rate? Ie, what interest rate will make consumers want to consume their entire endowment in every period. This is because consumers are identical and if one wants to borrow, then they all want to borrow and this would not be a equilibrium market clearing situation. So the only market clearing equilibrium that is possible involves no borrowing or lending so $c_t = y_t \; \forall \; t$. Thus,

$$\overline{R}_{t+1} = \frac{1}{\beta E_t[G_{t+1}]}.$$

$$\overline{R}_{t+1} = \frac{1}{\beta E_t[\left(\frac{c_{t+1}}{c_t}\right)^{-1}]}.$$

$$\overline{R}_{t+1} = \frac{1}{\beta E_t[\left(\frac{y_{t+1}}{y_t}\right)^{-1}]}.$$

$$\overline{R}_{t+1} = \frac{1}{\beta y_t E_t[\left(\frac{1}{y_{t+1}}\right)]}.$$

Note that since $y_t$ is iid,

$$E_t[\frac{1}{y_{t+1}}] = K \neq \frac{1}{\mu_t}.$$

Thus,

$$\overline{R}_{t+1} = \frac{1}{K\beta y_t}.$$

So the riskless interest rate is counter-cyclical in an endowment economy with iid shocks and no idiosyncratic shocks.

- Intuition: If an adverse shock ($y$ falls) occurs in the aggregate economy, people will want to borrow to smooth consumption (they will want to issue bonds). But this means bond prices will fall (ie, implicit interest rates will rise). If you pay $1000 for a bond with a coupon of $100 then your return initially is 100/1000 = 10%. But if the price of the bond now falls to $800, your return is 100/800 = 12.5%. So holders of bonds want the price to fall (they could sell before maturity and make a profit), while issuers want the price to rise.

- If there is a positive income shock ($y$ rises), everyone wants to save to smooth consumption so they all want to buy bonds. The bond price would be pushed up and thus the implicit interest rate would fall.

- In reality, the risk free interest rate is pro-cyclical so there must be another side to the story (investment) which overwhelms the consumption side.

**Application 2: Risky Interest Rate**

- Consider the return on a risky asset, $R_{t+1}$ and define the risk premium as:

$$E_t[R_{t+1} - \overline{R}_{t+1}].$$

Plugging in the euler equation from the C-CAPM model:

$$E_t[R_{t+1}] - \overline{R}_{t+1} = \frac{\frac{1}{\beta} - cov_t(R_{t+1}, G_{t+1})}{E_t[G_{t+1}]} - \frac{1}{\beta E_t[G_{t+1}]}.$$

$$E_t[R_{t+1}] - \overline{R}_{t+1} = \frac{1 - \beta cov_t(R_{t+1}, G_{t+1})}{\beta E_t[G_{t+1}]} - \frac{1}{\beta E_t[G_{t+1}]}.$$

$$E_t[R_{t+1}] - \overline{R}_{t+1} = \frac{1 - \beta cov_t(R_{t+1}, G_{t+1}) - 1}{\beta E_t[G_{t+1}]}.$$

$$E_t[R_{t+1}] - \overline{R}_{t+1} = \frac{-cov_t(R_{t+1}, G_{t+1})}{E_t[G_{t+1}]}.$$

- So when will the risky asset have a positive risk premium?

$$E_t[R_{t+1}] - \overline{R}_{t+1} > 0$$

$$\iff$$

$$-cov_t(R_{t+1}, G_{t+1}) > 0$$

$$\iff$$

$$cov_t(R_{t+1}, G_{t+1}) < 0$$

$$\iff$$

$$cov_t(R_{t+1}, \frac{u'(c_{t+1})}{u'(c_t)}) < 0$$

$$\iff$$

$$cov_t(R_{t+1}, \frac{u(c_{t+1})}{u(c_t)}) > 0$$

This last implication is true for concave utility functions. So the risk premium will be positive if the risky return is positively correlated with consumption growth.

- Intuition 1: If the the covariance is positive, then the asset pays off the most when consumption growth is high, ie when $u'(c_{t+1})$ is low (we have a lot of consumption and times are good). So it pays off the most when you need it the least. For an asset with a positive covariance, it might be called a "Fair-Weather-Friend."
  - All else equal, consumers would prefer a riskless asset that pays off equally in all states. All else equal, the price of a riskless asset will be higher than the price of an asset with positive covariance so $\overline{R}_{t+1} < E_t[R_{t+1}]$.
- Intuition 2: If the the covariance is negative, then the asset pays off the most when consumption growth is low, ie when $u'(c_{t+1})$ is high (we have very little consumption and times are bad). So it pays off the most when you need it the most. For an asset with a negative covariance, "A-Friend-In-Need-Is-A-Friend-Indeed."
  - All else equal, the price of a riskless asset will be lower than the price of an asset with negative covariance so $\overline{R}_{t+1} > E_t[R_{t+1}]$.
  - Insurance is a good example. Possibly gold.

# 14 Lecture 14: October 18, 2004

## 14.1 C-CAPM Asset Pricing Model

- Recall the equation from last time for this model:

$$E_t[R_{t+1}] - \overline{R}_{t+1} = \frac{-cov_t(R_{t+1}, G_{t+1})}{E_t[G_{t+1}]}.$$

  Note that the variance of the return does not enter the risk premium equation. Only the covariance. So, in general a more volatile asset will not necessarily generate a large (or even positive) risk premium.

- Unconditional risk premium:

$$E[R_{t+1}] - \overline{R}_{t+1} \approx \frac{-cov(R_{t+1}, G_{t+1})}{E[G_{t+1}]}.$$

  Though we technically cannot separate things like this, this should still hold approximately.

- In the data, Kocherlakota uses annual US data from 1880-1978 and estimates:

$$E[R^{stock}] = 1.07.$$

$$E[R^{treasury}] = 1.01.$$

$$Corr(R^{stock}, (\frac{c_{t+1}}{c_t})) > 0.$$

  So we get a risk premium of about 6% and since the correlation is positive, we should be able to use the C-CAPM model to predict the positive risk premium.

- Assuming that the C-CAPM model applies to aggregate data (it doesn't), requires that either we have quadratic utility, or complete markets. We need household level consumption to track aggregate consumption. Assume CRRA utility so that:

$$G_{t+1} = \left(\frac{c_{t+1}}{c_t}\right)^{-\alpha}.$$

  Where $\alpha$ is the coefficient of relative risk aversion, (RRA).

- For $\alpha = 1$, $u(c) = log(c)$ and:

$$Equity\ Premium(EP) = \frac{cov(R^{stock}, \frac{c_{t+1}}{c_t})}{E(c_{t+1}/c_t)^{-1}}.$$

  The denominator is estimated to be around 1. Decompose the covariance into:

$$cov(R^{stock}, \frac{c_{t+1}}{c_t}) = Corr(R^{stock}, c_{t+1}/c_t) * SD(R^{stock}) * SD(c_{t+1}/c_t).$$

Using annual data, Kocherlakota finds:

$$cov(R^{stock}, \frac{c_{t+1}}{c_t}) = 0.4 * 0.166 * 0.036 = 0.24\%.$$

Definitely NOT six percent!

- But that all assumed $\alpha = 1$. The standard deviation of consumption growth is increasing in $\alpha$ so we need a utility function which is highly curved to increase the risk premium. Need a lot of risk aversion!
- What does $\alpha$ have to be to generate a RP = 6%? $\alpha = 20$ !! Not likely. In a choice between two lotteries where in one case you get a 50/50 shot at $50,000$ or $100,000$ and in the other you get $X$ with certainty. If $\alpha = 20$, the agent would have to indifferent between the lottery and a sure thing of $51,858$. This seems unrealistic since most experiments find that people's risk aversion parameter is around 3 or 4.

**Equity Premium Puzzle (EPP)**

- For a realistic $\alpha$, the C-CAPM applied to aggregate data cannot explain the magnitude of the EP.
- Intuition: Aggregate consumption is too smooth and under complete markets, household consumption is also smooth. Both imply that unless $\alpha$ is very high, $u'(c)$ is fairly smooth. In other words, if $u'(c)$ is very smooth, then if the EP = 6 percent, it is optimal to borrow a large amount risk-free and invest more than 100 percent of your wealth in stocks. This would drive down the EP.
- The flip side to the EPP is the Risk-Free Rate Puzzle (RFRP). For $\alpha = 20$, the required risk free rate implied by the C-CAPM model is around 18 percent !! Under uncertainty:

$$E(\frac{c_{t+1}}{c_t}) = [\beta E(\overline{R})]^{1/\alpha}.$$

Where $1/\alpha$ is the intertemporal elasticity of substitution: $IES = 1/RRA = 0.05$. For $\beta \leq 1$, this is where the 18 percent comes from because the LHS is approximately 1.018 percent. If $\alpha$ is high, IES is low (assuming time separable utility) and we need a high risk free rate to tolerate around 2 percent consumption growth over time.
- If we allow for uncertainty, we do a bit better, since $u''' > 0$ implies higher optimal consumption growth for any given level of $\alpha$, $\beta$, and the risk free rate (Precautionary Savings). But in aggregate data, consumption is smooth so that PS motive is weak even for a high $\alpha$.

**Solutions to the Equity Premium Puzzle**

- (1) Alternative Preferences.

* (A) Kocherlakota assumes $\beta > 1$ to explain the high consumption growth without invoking a high risk free rate. People would have to value the future a lot. Not likely.
* (B) Weil tries non-separable preferences. This allows for $IES \neq 1/RRA$. Could still have $\alpha = 20$ but not such a low IES.
* (C) Cambell and Cochrane try Habit formation. Assume utility is of the form:

$$u(c) = (c - c^{hab})^{1-\alpha}.$$

  With:

$$c^{hab} = \gamma[\frac{1}{T}\sum_{i=1}^{T} c_{t-1}], \quad \gamma \in (0,1).$$

  So our habitual consumption is some fraction of our past levels of consumption. This can explain the EP in aggregate data with a much lower $\alpha$. See G-14.1. For CRRA utility, aggregate consumption is large so we are on a fairly flat part of both the $u(c)$ and $u'(c)$ curves. So $SD(c_{t+1}/c_t)$ is low. But $SD((c_{t+1} - c^{hab})/(c_t - c^{hab}))$ will be much higher. This might drive the higher risk premium. So for a given $\alpha$, we feel more risk averse under habit formation. In reality, this is NOT consistent with lab experiments which suggest that people do not feel risk averse. They do not behave as if they were close to 0 in terms of consumption such that the variation in their consumption growth was high.
* (D) Thaler and Benartzi (1995 QJE) have a model of myopic loss aversion. They explain the six percent RP if people are loss averse and if they focus only on the 1 year rate of return on assets and not multi-year returns. This was just preliminary stuff though. More work needs to be done on this.

– (2) Incomplete Markets.

* Suppose the C-CAPM holds at the household level but fails at the aggregate level (due possibly to incomplete markets). Then we could potentially explain the EP for lower values of $\alpha$ because individual consumption growth is more volatile than aggregate consumption growth.
* The problem with this is the lack of reliable household level consumption data. However, even in theory, Deaton and Zeldes for example looked at what ever consumption data we do have and did not find enough volatility to explain the high EP. Individual saving and dissaving generates a lot of consumption smoothing.

– (3) Asset Market Frictions.

* This impiles a failure of the LCH/C-CAPM euler equation at the household level. Mankiw and Zeldes find that in the 1984 PSID, only 27.6 percent of households actually own stocks. Why not?
  · Households are liquidity constrained.
  · There are short-sale constraints.
  · Trading costs.

* So they go ahead and calculate the EP using only households that hold stock and create aggregate consumption data only for stock holders. They find that the correlation between the return on stocks and consumption growth, along with the standard deviation of consumption growth are higher! To generate an EP of 6 percent you only need a coefficent of relative risk aversion of around 4 or 5. Much more reasonable.

# Midterm Review

## 14.2 Models

### LHC under certainty

- 2 motive which cause tension: (1) Intertemporal Substitution (IS) - a high interest rate implies high consumption growth because the relative price of $c_2$ has fallen. Defer lots of consumption until tomorrow. (2) Consumption Smoothing (CS).
- Power utility: $u(c) = \frac{\sigma}{\sigma - 1} c^{(\sigma-1)/\sigma}$. Implies:

  $$\frac{c_2}{c_1} = [\beta(1+r)]^{\sigma}.$$

  Where, $\sigma$ is the intertemporal elasticity of substitution and:

  $$\sigma = \frac{d(ln(c_2/c_1))}{d(ln(1+r))}.$$

  As $\sigma \to \infty$, $u(c) \to$ linear and IS dominates CS. No impetus for CS. As $\sigma \to 0$, $u(c)$ is highly curved and CS dominates IS. Less tilting of the consumption path.
- Under certainty, no correlation between $\Delta c$ and $\Delta y$.
- Euler equation under certainty:

  $$u'(c_t) = \beta(1+r)u'(c_{t+1}).$$

### LHC under uncertainty

- Under uncertainty (Rational Expectations), define:
  The additive risk premium as the p that satisfies:

  $$E[u(c+e)] = u(c + E(e) - p).$$

  $$\Longrightarrow p \approx \frac{1}{2}\sigma_e^2 \underbrace{\left(-\frac{u''(c)}{u'(c)}\right)}_{Coeff\ Absolute\ Risk\ Aversion}.$$

  The multiplicative risk premium as the p that satisfies:

  $$E[u(ce)] = u(cE(e) - pc).$$

  $$\Longrightarrow p \approx \frac{1}{2}\sigma_e^2 \underbrace{\left(-\frac{u''(c)c}{u'(c)}\right)}_{Coeff\ Relative\ Risk\ Aversion}.$$

- Link between risk aversion motive the consumption smoothing motive:

* For LOW curvature $u(c)$ ($\sigma$ high in Power), impiles both weak CS (high IES) and a willingness to bear risks. So there is a willingness to tolerate consumption variation over TIME and STATES OF NATURE.
* For HIGH curvature $u(c)$ ($\sigma$ low in Power), impiles both strong CS (low IES) and an unwillingness to bear risks. So there is an unwillingness to tolerate consumption variation over TIME and STATES OF NATURE.

– Markov Uncertainty: $\pi_{t+1}(x_{t+1}|x^t) = \pi(x_{t+1}|x_t)$.

– No Ponzi Games: debt must grow slower than the interest rate in the limit:

$$lim_{L\to\infty} a_L \prod_{t=1}^{L} \frac{1}{1+r_t} = 0.$$

– Time consistency: A decision rule is time consistent if the agent could reoptimize in every period, he would still choose the same DR.

– Hyperbolic Discounting:

$$u(c_0) + \alpha E_0 \sum_{t=1}^{T} \beta^t u(c_t), \quad 0 > \alpha \leq 1.$$

– Solving Bellman's: Guess and Verify, Backwards Iteration, Solve for properties without finding actual form.

– Euler equation under uncertainty:

$$u'(c_t) = \beta(1 + r_{t+1})E_t u'(c_{t+1}).$$

**LHC - Quadratic Utility**

– Quadratic Utility Case: $u(c) = c_t - \frac{a}{2}c_t^2 \implies c_t = E_t[c_s] \ \forall \ s > t$. Using IBC, we find the closed form DR:

$$c_t = \frac{r}{1+r}(a_t + h_t).$$

If we assume $y_t = \alpha y_{t-1} + \epsilon_t$ (AR1). $E_t[y_{t+k}] = \alpha^k y_t$. If $\alpha = 0$, income is white noise and shocks are transitory. If $\alpha = 1$, income is a random walk and shocks are permanent.

– Under uncertainty with quadratic utility, AR1 income we find that $\Delta c_t$ should be uncorrelated with predictable income changes and $\Delta c_t$ should be positively correlated with innovations to income (with strong effects, the larger is $\alpha$).

– When can we aggregate from household level decisions to the economy as a whole (quadratic utility or complete markets endowment economy).

* Complete Markets. At time 0, households can buy or sell a complete set of state-contingent Arrow-Debreu Securities. Pays off if a certain history is realized. Assume Zero net supply. For every claim, there is an obligation. Static problem. So result:

- (1) All $c_t^i$ are perfectly correlated with aggregate consumption.
- (2) Households have full insurance against all idiosyncratic shocks. A households consumption depends on aggregate income but not on it's own income. Household simply buys or sell arrow-debreu's in good or bad states to smooth consumption.
- (3) In a complete markets endowment economy, consumption aggregates up so a rejection of the LCH at the aggregate level implies a household level rejection as well.
- (*) $\Delta c^i$ will be correlated with innovations to $\Delta y^i$, but only the non-idiosyncratic component. $\Delta c^i$ $\Delta c^i$ will also be correlated with $\Delta y$ (aggregate) so there is NOT full insurance against everything. Finally $\Delta c^i$ may be correlated with the idiosyncratic component of $\Delta y^i$ ONLY if there are taste shocks.

Complete markets will fails if (1) there is hidden information or (2) there is limited enforcement.

**Loss Aversion**

– Utility is of the form:

$$u(c) = V(r) + W(c - r).$$

**Eliminating Certainty Equivalence: Non-negativity Constraints**

– Certainty Equivalence: Optimal behavior in the model is identical to behavior inan otherwise identical model in which future random variables are assumed to take on their mean values. The combination of nonnegativity constraints on consumption and a solvency condition provides for a Natural Borrowing Constraint.

$$a_t \geq -\sum_{s=t}^{T}(1+r)^{-(s-t)} * min(y_s).$$

**Eliminating Certainty Equivalence: Non-quadratic Utility**

– Suppose utility takes the form:

$$V = u(c_0) + E_0[u(c_1)].$$

If the third derivatives of utility is greater than zero, this means that u' is a convex function and consumers will save today to avoid possible losses tomorrow (Precautionary Savings).

– Jensen's Inequality: $E[u'(c_{t+1})] > u'(E[c_{t+1}])$ implies:

$$c_t < E_t[c_{t+1}].$$

**Precautionary Savings**

- Under CRRA utility, if we assume that consumption is log normal, we find a precautionary savings motive. We can also find this taking a 2nd order taylor series expansion. Consumption growth is increasing in the variance of consumption growth and in the risk aversion parameter. Meaning we will save more today.

**Asset Pricing**

- Key Equation:
$$E_t[R_{t+1}] = \frac{1/\beta - cov_t(R_{t+1}, G_{t+1})}{E_t[G_{t+1}]}.$$
- Riskless interest rate is counter-cyclical. Note true in the real world.
- Positive risk premium if:
$$cov_t(R_{t+1}, \frac{u(c_{t+1})}{u(c_t)}) > 0.$$
- So:
$$P(FWF) < P(RISKLESS) < P(FNFI).$$

## 14.3 Articles

### Hall 1978 JPE

- Test LCH using Aggregate Data.
- Assume: quadratic utility, $\beta(1+r) = 1$, no liquidity constraints, utility separable (durable/nondurable).
- NIPA Quarterly 1948-1977.
- Regression:
$$c_{t+1} = \alpha_0 + \alpha_1 c_t + \beta z_t + \epsilon_{t+1}.$$
  Where $z_t$ is a vector of variables in the information set at time $t$.
- Null: LCH true: $\alpha_0 = 0$, $\alpha_1 = 1$, $\beta = 0$.
- Result: Fail to reject for $z_t$ = income and consumption lags, but reject for $z_t$ = real $S\&P$ 500 index.
- Critique: Type I errors if $\beta(1+r) \neq 1$. Consumption should drift up if greater than one. Omitted Variable Bias if $r_t$ should be included in regression with stock prices. Power of test also poor. We would like to select optimal $z_t$'s such that we have the power to reject plausible alternative models. Hall's third regression has the most power so since he finds evidence against the LCH in that regression, we should put the most weight on this result.

- Since Hall used Quadratic utility, consumption should aggregate up from the household level so his rejection of the LCH at the aggregate level should imply a rejection at the household level as well.

**Wilcox 1989 JPE**

- Aggregate data to test LCH.
- Used changes in the social security benefit since these are announced several months in advance.
- Adjustments to consumption should be incorporated at the annoucement, not at the effective date.
- Durable goods may much at the announcement but if anything, should fall at the effective.
- Result: Positive bias for durable spending at annoucement. Inconsistent with LCH.
- Critique: Liquidity constraints may be part of the reason. Agents cannot spend the SS increase untill they have it because they are constrained.

**Cochrane 1991 JPE**

- Household level test of complete markets hypothesis: Are households fully insured against idiosyncratic risks?
- Data: PSID, 4629 households and a subsample of 1741 with no composition changes, 1980-1983.
- Model:

$$\Delta c_i = \alpha + \beta x_i + \epsilon_i.$$

- Null: Complete markets correct: $\beta = 0$. (Where $x_i$ is a measure of idiosyncratic household shocks).
- Result: reject $\beta = 0$ when $x_i$'s are involuntary job loss and illness, but not for strike days.
- Critique: Noisey consumption data in PSID (TII error). Taste shocks may bias $\beta$ away from zero (T1 error). However, consumption data looks ok and use the subsample to avoid the second issue.
- Must use household level data to test the LCH. No aggregation.

**Zeldes 1989 JPE**

- Household level test of Liquidity Constraints in the LCH. LCH should hold for unconstrained households, but fail for those who cannot borrow today to smooth consumption.
- Data: PSID households. 1968-1982. 4000 households.

– Model:

$$\Delta log(c_{it}) = \alpha_{i0} + \alpha_1 (AGE)_{i,t-1} + \alpha_2 (Food)_{it} + \frac{1}{\rho} log(1 + r_{it}) + \lambda y_{i,t-1} + \epsilon_{it}.$$

– Estimate two samples for rich and poor.
– Null: LCH true: $\lambda = 0$ for both samples. Alternatives: Liquidity Constraints: $\lambda = 0$ for rich, $\lambda \neq 0$ for poor. Keynesian Myopia: $\lambda \neq 0$.
– Result: Using non-housing wealth, we get a liquidity constraints result. Using housing wealth, we get a Myopia result.
– Critique: Since a bunch of households have ONLY housing wealth, this biases the coefficients in the second split. Non-housing wealth would be the correct split and hence we would not reject the LCH if we include the LC caveat.
– Shea also says this might just be heterogeneity in people's ability to plan ahead. If people are more prudent about planning, they would be less likely to violate the LCH then those that do not plan, irrespective of wealth. So Zeldes results could be indistinguishable from myopia.

**Shea 1995 AER**

– Household level test using a good measure of $E_{t-1}\Delta y_t$. EDWage: expected percentage change in real wages.
– See notes.
– Results: No evidence for LCH, LC, or keynesian myopia.

**Zeldes 1989 QJE**

– Test the precautionary savings motive under CRRA utility.
– Consumption function turns out to be concave and people do save more if they are more uncertain about the future.

**Carroll 1997 QJE**

– The variance of consumption is likely to be time variant. In particular, it is likely to be a decreasing function of cash on hand. –This is backed up by the concavity of the consumption function.

**Deaton 1991 Econometrica**

– Zeldes predicts that people should save a lot, but in reality we don't see that? Maybe social insurance (Hubbard) or Impatience (Deaton) reduces savings today?
– Consumption at the household level turns out to be very smooth. Assets are very low. Consumption volatility is asymmetric. Spikes down in bad times. Positive shocks however are saved.

# 15 Lecture 15: October 25, 2004 - Drazen Start

## 15.1 Basic Growth Models

- Consider policy question number 1: How does the world bank calculate the required capital investment in a developing country to sustain growth?
- Suppose the target growth is 3%, population growth is 1% and we therefore need output to grow at 4%.
- Ouput growth is proportional to investment as a percentage of GDP so 4% growth requires 12% investment if there is a 3:1 $K/Y$ ratio. Add in 1% depreciation and we get a required 13% investment rate.
- This all seems to make sense but in actuality there is more going on.
- The WB would finish the analysis noting that savings was 5% of GDP in the country so this leads to a $13 - 5 = 8\%$ gap which should be financed by foreign aid. There are of course problems with this: one being that foreign aid will not always be designated to investment.

**The Solow Growth Model**

- I. Technology. Assume the production function take the form:

$$Y_t = F(K_t, L_t). \quad (1)$$

Assume:

  - (1) $F$ is $C^2$, $F_K > 0, F_{KK} < 0, F_L > 0, F_{LL} < 0$.
  - (2) $F$ is hod(1) so $F(\lambda K, \lambda L) = \lambda F(K, L)$.
  - (3) $F$ is increasing in $K$ and $L$.
  - (4) Each input is strictly necessary: $F(0, L) = F(K, 0) = 0$.
  - (5) $F(K, 1)$ is strictly concave.
  - (6) Inada conditions satisfied: $lim_{K\to 0} F_K(K, 1) = \infty$, $lim_{K\to\infty} F_K(K, 1) = 0$.

- II. Inputs. Assume labor is supplied inelastically so all labor is used. If population grows at a rate $n$, then:

$$L_{t+1} = (1 + n)L_t. \quad (2)$$

Assume a closed economy so that gross investment equals savings which occurs at a CONSTANT rate, $s$. This is equivalent to saying the MPC is constant. Net investment equals gross investment less depreciation. Thus:

$$K_{t+1} = sY_t + (1 - \delta)K_t. \quad (3)$$

$$K_{t+1} - K_t = sY_t - \delta K_t.$$

- III. Evolution of the Economy. Use the state variable $k_t = \frac{K_t}{L_t}$. Divide (3) by (2):

$$\frac{K_{t+1}}{L_{t+1}} = \frac{sY_t + (1-\delta)K_t}{(1+n)L_t}.$$

$$k_{t+1} = \frac{s}{1+n}\frac{Y_t}{L_t} + \frac{1-\delta}{1+n}k_t.$$

Note that:

$$y_t = \frac{Y_t}{L_t} = \frac{1}{L_t}F(K_t, L_t) = F(k_t, 1) \equiv f(k_t).$$

So $f(k_t)$ is the intensive production function (in per capita terms). So,

$$k_{t+1} = \frac{s}{1+n}f(k_t) + \frac{1-\delta}{1+n}k_t \equiv g(k_t) \quad (4).$$

We set the RHS equal to $g(k_t)$ and then plot in G-15.1 We know that $g(0) = 0, g'(k) > 0$ and since $f$ is concave, $g$ will be concave. Thus the steady state in this picture is where $k_t = k_{t+1}$ which is clear from the picture that it occurs at $k_t = 0$ and $k_t^{ss}$. Solving for the positive SS,

$$k^{ss} = \frac{s}{1+n}f(k^{ss}) + \frac{1-\delta}{1+n}k^{ss}.$$

$$\frac{f(k^{ss})}{k^{ss}} = \frac{n+\delta}{s}. \quad (5)$$

Equation 5 is the Harrod-Domar condition which says that the output to capital ratio is constant in the steady state. In pre-Solovian growth theory, Domar "may" have said that this SS is unstable and very infrequently reached, but Solow has shown that this is not the case. In the appendix we show that:

$$lim_{k\to 0}\frac{f(k)}{k} = \infty,$$

$$lim_{k\to \infty}\frac{f(k)}{k} = 0,$$

$$\frac{f(k)}{k} \in (0, \infty), \quad \frac{n+\delta}{s} \in (0, \infty).$$

Thus there will exist a $k$ which satisfies equation 5. So the output to capital ratio must be constant in the steady state though this will depend on the values of $n$, $\delta$, and $s$.

- IV. Properties of the Steady State. See G-15.2. Note that using the 45 degree line method, it is clear that $k^{ss}$ is stable and the sufficient condition for this stability is concavity of the production function. This implies that:

$$g'(k^{ss}) < 1,$$

ie $g(\cdot)$ cuts the 45 degree line from above at the steady state. It's also clear from the graphs that this implies:

$$f'(k^{ss}) < \frac{n + \delta}{s}.$$

Or substituting on the RHS:

$$f'(k^{ss}) < \frac{f(k^{ss})}{k^{ss}}.$$

This is guaranteed by the strict concavity of $f(\cdot)$.

- Thus balanced growth is likely and convergent and we don't have the paradox suggested by Harrod-Domar. The results of Solow are somewhat disappointing though based on what we see in reality:
  - (1) Constant $f(k)/k$. This actually grows exponentially.
  - (2) No technical progress in the model. This is actually an important engine of growth.
  - (3) Choice of savings in NOT endogeneous. Clearly this should be in the model.

# 16 Lecture 16: October 27, 2004

## 16.1 Overlapping Generations Model

- Samuelson (1958) introduced this model without capital and Diamond (1965) added capital and government debt. The question is: "What is the burden of the debt on future generations?"
- If the debt is internal, then repaying the debt is simply a matter of transfering money within our own economy. However, external debt is a different story.

**Samuelson (1958)**

- Starting with the Samuelson model. Consider an endowment economy where each individual lives for two periods: young ($y$) and old ($o$). There is no capital so all loans finance consumption. Individuals receive an endowment of chocolate in period 1 ($e^y = 1$ and $e^0 = 0$). The endowment is perishable. Optimal consumption would be to consume half the chocolate when young and half when old but since it is not storable, this is not feasible.
- Introduce overlapping generations. Assume population growth, $n = 0$, and $L = 1$ (supply of labor in period 1 is 1 which nets a single chocolate endowment). An individual in generation $t$ lives from period $t$ to $t + 1$ where he is young in the first, old in the second. In period $t+1$, a new generation is born and lives to period $t+2$, when a new generation is again born. And so on. Notice that at time $t + 1$, optimal consumption for two individuals is 1 and the total endowment available between the two is also 1 so the optimal consumption path is at least feasible. But how does Mr. $t + 1$ know that Mr. $t+2$ will give Mr. $t+1$ a chocolate in period $t+2$ in exchange for Mr. $t+1$ giving Mr. $t$ a half a chocolate in period $t+1$ ? Well, he doesn't unless there are social contracts.
- Samuelson introduces an asset $= M^s =$ \$1. So there is this single dollar bill in the economy now that the old individual initially has. Now Mr. $t$ pays Mr. $t + 1$ a dollar for his half a chocolate and then Mr. $t + 1$ pays Mr. $t + 2$ the dollar for his half in period $t+2$, and so on. But is the model much different from a social contracts story? Not really. The young guy should only accept the dollar for his chocolate if he thinks the next generation will also accept his dollar. This idea is very similar to a social contract.

**Diamond 1965**

- **National Income Accounting**
- Production function:

$$Y_t = F(K_t, L_t),$$

which is assumed to be CRS and $K_t$ must be set aside in the previous period for use in the next period. Capital accumulation equation:

$$F(K_t, L_t) + (1-\delta)K_t = C_t + K_{t+1}.$$

If there are $L_t$ young individuals in period $t$ and $L_{t-1}$ old individuals in period $t$, then total consumption is:

$$L_t C_t^y + L_{t-1} C_t^o.$$

Substituting and rearranging:

$$F(K_t, L_t) - (K_{t+1} - (1-\delta)K_t) = L_t C_t^y + L_{t-1} C_t^o. \quad (1)$$

Gross investment is:

$$K_{t+1} - K_t + \delta K_t.$$

Net investment equals gross investment less depreciation:

$$K_{t+1} - K_t.$$

In the steady state, all variables grow at the same rate:

$$K_{t+1} = (1+n)K_t,$$

$$L_{t+1} = (1+n)L_t.$$

And this implies:

$$k_t = \frac{K_t}{L_t} = k_{t+1} = k.$$

Divide (1) by $L_t$:

$$f(k_t) - \frac{(K_{t+1} - (1-\delta)K_t)}{L_t} = C_t^y + \frac{L_{t-1}}{L_t} C_t^o.$$

$$f(k_t) - \frac{(1+n)K_{t+1}}{L_{t+1}} + \frac{(1-\delta)K_t}{L_t} = C_t^y + \frac{L_{t-1}}{(1+n)L_{t-1}} C_t^o.$$

$$f(k_t) - (1+n)k_{t+1} + (1-\delta)k_t = C_t^y + \frac{C_t^0}{1+n}.$$

Evalute at the steady state:

$$f(k) - (1+n)k + (1-\delta)k = C^y + \frac{C^0}{1+n}.$$

$$f(k) - (n+\delta)k = C^y + \frac{C^0}{1+n}. \quad (2)$$

- Now, does the market lead to optimal investment? What level of capital optimizes consumption? Consider maximizing the LHS of equation 2:

$$Max_k\ f(k) - (n+\delta)k \Longrightarrow f'(k) = n + \delta \longrightarrow \text{Golden Rule.}$$

Thus,

$$k^{GR} = (f')^{-1}(n+\delta),$$

maximizes the consumption possibilities. Note that it does NOT depend on the utility function of the individuals. It does not take into account the cost of getting to the golden rule, it's just an optimum.

- **Competitive Economy**
- We need information about factor markets to go further. In a competitive economy, the wage rate equals the marginal product of labor and the net return on capital equals the marginal product less depreciation. So:

$$w_t = F_{L_t}.$$

$$r_t = F_{K_t} - \delta.$$

With a CRS (hod(1)) production function, we have the following euler equation:

$$F(K_t, L_t) = F_{K_t}K_t + F_{L_t}L_t. \quad (3)$$

Substitute in for $w_t$ and $r_t$:

$$F(K_t, L_t) = (r_t + \delta)K_t + w_tL_t.$$

$$F(K_t, L_t) - \delta K_t = r_tK_t + w_tL_t. \quad (4)$$

Divide (4) by $L_t$:

$$f(k_t) - \delta k_t = r_tk_t + w_t.$$

$$w_t = f(k_t) - \delta k_t - r_tk_t.$$

But recall from the Solow Model, $F_K(K_t, L_t) = f'(k_t)$, but $r_t = F_K - \delta$, So $r_t = f'(k_t) - \delta$. Substituting this in:

$$w_t = f(k_t) - \delta k_t - (f'(k_t) - \delta)k_t.$$

$$w_t = f(k_t) - k_tf'(k_t).$$

So in the end, we have two equations relating the return on labor and capital to $k_t$:

$$r_t = f'(k_t) - \delta. \quad (5)$$

$$w_t = f(k_t) - k_tf'(k_t). \quad (6)$$

Thus consider $f(k) = k^{\alpha}$. So $f'(k) = \alpha k^{\alpha-1}$. So,

$$(5) \Rightarrow r = \alpha k^{\alpha-1} - \delta \Longrightarrow k = (\frac{r+\delta}{\alpha})^{1/(\alpha-1)}.$$

$$(6) \Rightarrow w = k^{\alpha} - k\alpha k^{\alpha-1}.$$

$$\Longrightarrow w = k^{\alpha} - \alpha k^{\alpha} = k^{\alpha}(1-\alpha).$$

$$\Longrightarrow k = (\frac{w}{1-\alpha})^{1/\alpha}.$$

Setting equal:

$$(\frac{w}{1-\alpha})^{1/\alpha} = (\frac{r+\delta}{\alpha})^{1/(\alpha-1)}.$$

$$\frac{w}{1-\alpha} = (\frac{r+\delta}{\alpha})^{\alpha/(\alpha-1)}.$$

$$w = (1-\alpha)(\frac{r+\delta}{\alpha})^{\alpha/(\alpha-1)}.$$

And in general:

$$w_t = \phi(r_t), \quad with\ \phi' < 0.$$

This is the factor price fontier.

- **Optimal Consumption**
- Agents maximize:

$$max_{c_t^y, c_{t+1}^o}\ u(c_t^y, c_{t+1}^o),$$

subject to:

$$w_t = c_t^y + s_t,$$

$$c_{t+1}^0 = (1 + r_{t+1})s_t.$$

Where $s_t$ is savings at time $t$. This yields an optimal savings rate of $s_t = s(w_t, r_{t+1})$. Note that investment equals savings of the old and the young. Aggregate savings of the young at period $t$ is:

$$L_t s_t.$$

The old have income in period $t$ equal to $s_{t-1}r_t$ and they consume $(1+r_t)s_{t-1}$ so there is aggregate dissavings among the old (they do not save any more – they consume interest on their savings from the previous period as well as the savings itself). Dissavings equals $s_{t-1}$ and aggregate dissavings is:

$$L_{t-1}s_{t-1}.$$

Hence net aggregate savings is:

$$L_t s_t - L_{t-1}s_{t-1}.$$

Since net investment in period $t$ is $K_{t+1} - K_t$, we have:

$$L_t s_t - L_{t-1}s_{t-1} = K_{t+1} - K_t.$$

And this is just a first order difference equation with one solution being: $K_{t+1} = L_t s_t$. This is easy to see if you substitute on the LHS. Note that in Drazen's notes he has the timing off by one. Capital is only realized in the following period so net investment in period $t$ is equal to investment in the following period less today's investment. Correcting this yields the more "obvious" result: $K_{t+1} = L_t s_t = L_t s(w_t, r_{t+1})$. The old people dissave everything and next period capital is equal to only the savings of the young. Rewrite the last equation as:

$$\frac{K_{t+1}}{L_t} = s_t.$$

$$\frac{L_{t+1}}{L_{t+1}}\frac{K_{t+1}}{L_t} = s(w_t, r_{t+1}).$$

$$\frac{L_{t+1}}{L_t}k_{t+1} = s(w_t, r_{t+1}).$$

$$(1+n)k_{t+1} = s(w_t, r_{t+1}).$$

Substitute in from equations (5) and (6):

$$(1+n)k_{t+1} = s(f(k_t) - k_t f'(k_t), f'(k_{t+1}) - \delta).$$

Implicitly, this equals:

$$k_{t+1} = A(k_t), \quad given\ k_0 = \frac{K_0}{L_0}.$$

A first order difference equation. See G-16.1 for the phase diagram. Note that $A(0) = 0$, $lim_{k_t \to \infty} A(k_t)$ implies $k_t > k_{t+1}$ and $A'(k_t) = \frac{dk_{t+1}}{dk_t} > 0$. The stable equilibriums occur when $A'(k) < 1$.

- In general, the stable equilibrium of this problem will not equal the golden rule level of capital. We may even end up with more capital then the golden rule which is called the Dynamic Inefficiency. There would be a pareto improving outcome that this model does not predict. Why? Externalities? While there may not seem to be any obvious externality here, the essential one is that every generation's decisions effect the outcome in all future generations, while the current generation does NOT take these effects into consideration when choosing how much to save. This is the externality and the result is a dynamic inefficiency.

# 17 Lecture 17: November 1, 2004

## 17.1 Overlapping Generations Model

- Recall the specific model from last time: $f(k) = k^\alpha$ and $u(\cdot) = \gamma ln(c_t^y) + (1-\gamma) ln c_{t+1}^o$. See homework for derivation of:

$$k_{t+1} = \frac{(1-\gamma)(1-\alpha)}{1+n} k_t^\alpha.$$

Solve for $k_t$:

$$k_t = \left( \frac{(1+n)}{(1-\gamma)(1-\alpha)} k_{t+1} \right)^{1/\alpha}.$$

$$f(k_t) = \frac{(1+n)}{(1-\gamma)(1-\alpha)} k_{t+1}.$$

SS:

$$f(k) = \frac{(1+n)}{(1-\gamma)(1-\alpha)} k.$$

$$f'(k) = \frac{(1+n)}{(1-\gamma)(1-\alpha)}.$$

Note if $\delta = 0$, $r_{t+1} = f'(k_{t+1})$. So at the SS,

$$r^{ss} = f'(k) = \frac{(1+n)}{(1-\gamma)(1-\alpha)} \neq n.$$

He has an extra $\alpha$ !!

- So if $r^{ss} > n$, $k^{ss} < k^{GR}$ and if $r^{ss} < n$, $k^{ss} > k^{GR}$ in which case we have a dynamic inefficiency ($k^{GR}$ pareto dominates.)

- What is the externality causing this market failure? See last lecture's notes. We can fix the problem by introducing optimization over an infinite horizon, or by introducing additional terms to the utility function:

$$u_t = \gamma ln(c_t^y) + (1-\gamma) ln(c_{t+1}^o) + \underbrace{\phi ln(bequest_t)}_{Warm\ Glow\ Function}.$$

$$u_t = \gamma ln(c_t^y) + (1-\gamma) ln(c_{t+1}^o) + \phi ln(utility\ of\ next\ generation).$$

**What is the effect of Government Debt?**

- Next we would like to investigate the effect of government debt on the capital stock, interest rates, and overall welfare.

- In comparing fiscal policies we can use one of two methods.

- (1) Balanced Budget Incidence - Keep deficit constant.
- (2) Differential Incidence - Keep expenditure constant and look at different ways of financing the expenditures.

- Diamond uses method 2 and assume that expenditures = 0.
- Suppose there is 1-period government debt such that all debt issued in period $t$ must be repaid in period $t+1$. Thus,

$$\underbrace{T_t}_{Taxes} = \underbrace{B_t(1+r_t) - B_{t+1}}_{Payments\ on\ t-1\ debt\ plus\ new\ debt}.$$

  The taxes the government raised in period $t$ must go to repaying last period's debt, but they can also issue new debt to pay off the old debt.

- In per capita terms (per young person specifically):

$$\tau_t = \frac{T_t}{L_t} = \frac{B_t}{L_t}(1+r_t) - \frac{B_{t+1}}{L_t}.$$

  Note that $L_{t+1} = (1+n)L_t$,

$$\tau_t = b_t(1+r_t) - (1+n)b_{t+1}.$$

  Net wages of a young person are:

$$\hat{w}_t = w_t - \tau_t.$$

- We will consider policies such that $B_{t+1} = (1+n)B_t$ so debt grows at the same rate as population growth. Assume $\delta = 0$. Thus:

$$b_t = b_{t+1} = b.$$

  Plugging into our equation for per captial taxation:

$$\tau_t = b(1+r_t) - (1+n)b = (r_t - n)b.$$

  Plugging into our net wage equation:

$$\hat{w}_t = w_t - \tau_t = w_t - (r_t - n)b. \quad (1)$$

**Who holds the Debt?**

- The debt could be held all externally (non-domestically) which yields an equilibrium condition:

$$L_t s(\hat{w}_t, r_{t+1}) = K_{t+1},$$

which is the result we have above. Savings goes completely to capital accumulation, and although taxes enter through the wages and the interest rate, all interest payments on the debt flow abroad. National income is lower compared to the GDP.

- If the debt is held completely internally, the equilibrium condition is:

$$L_t s(\hat{w}_t, r_{t+1}) = K_{t+1} + B_{t+1} \quad (2).$$

In this case, holding debt has both a direct effect on savings and also may crowd out private investment. $K_{t+1}$ will necessarily fall to increase $B_{t+1}$. Divide (2) by $L_t$:

$$s(\hat{w}_t, r_{t+1}) = (1+n)(k_{t+1} + b_{t+1}).$$

Substitute in from above for the wage and interest rate noting that $b$ will be constant in the steady state:

$$s(w_t - (r_t - n)b, f'(k_{k+1})) = (1+n)(k_{t+1} + b).$$

$$s\Big(f(k_t) - k_t f'(k_t) - (f'(k_t) - n)b, f'(k_{t+1})\Big) = (1+n)(k_{t+1} + b). \quad (3)$$

See G-17.1 which shows our original equilibrium when there is no government debt and then when $b > 0$. Note the $A$ function including debt is lower which is shown by considering the partial of $k_{t+1}$ wrt $b$ holding $k_t$ constant. Differentiate (3) wrt $b$.

$$\frac{\partial s}{\partial w} * (-(r_t - n)) + \frac{\partial s}{\partial r_{t+1}} * f''(k_{t+1}) * \frac{\partial k_{t+1}}{\partial b} = (1+n)(\frac{\partial k_{t+1}}{\partial b} + 1).$$

Rearrange:

$$(1+n)\frac{\partial k_{t+1}}{\partial b} + (1+n) = \frac{\partial s}{\partial w} * (-(r_t - n)) + \frac{\partial s}{\partial r_{t+1}} * f''(k_{t+1}) * \frac{\partial k_{t+1}}{\partial b}.$$

$$(1+n)\frac{\partial k_{t+1}}{\partial b} - \frac{\partial s}{\partial r_{t+1}} * f''(k_{t+1}) * \frac{\partial k_{t+1}}{\partial b} = \frac{\partial s}{\partial w} * (-(r_t - n)) - (1+n).$$

$$\frac{\partial k_{t+1}}{\partial b}(1 + n - \frac{\partial s}{\partial r_{t+1}} * f''(k_{t+1})) = \frac{\partial s}{\partial w} * (-(r_t - n)) - (1+n).$$

$$\frac{\partial k_{t+1}}{\partial b}(1 + n - \frac{\partial s}{\partial r_{t+1}} * f''(k_{t+1})) = -r_t\frac{\partial s}{\partial w} + n * \frac{\partial s}{\partial w} - (1+n).$$

$$\frac{\partial k_{t+1}}{\partial b}(1 + n - \frac{\partial s}{\partial r_{t+1}} * f''(k_{t+1})) = -r_t\frac{\partial s}{\partial w} + n * (\frac{\partial s}{\partial w} - 1) - 1.$$

$$\frac{\partial k_{t+1}}{\partial b}(1 + n - \frac{\partial s}{\partial r_{t+1}} * f''(k_{t+1})) = -\Big(1 + n(1 - \frac{\partial s}{\partial w}) + r * \frac{\partial s}{\partial w}\Big).$$

$$\frac{\partial k_{t+1}}{\partial b} = -\frac{\Big(1 + n(1 - \frac{\partial s}{\partial w}) + r * \frac{\partial s}{\partial w}\Big)}{(1 + n - \frac{\partial s}{\partial r_{t+1}} * f''(k_{t+1}))}.$$

- Note that $\partial s/\partial w$ is less than 1 and positive so the numerator is positive. If we assume the substitution effects dominate, $\partial s/\partial r > 0$ as well so the denominator is also positive. With the negative sign out front,

$$\frac{\partial k_{t+1}}{\partial b} < 0.$$

And hence the downward shift in the $A$ function in the graph.

- See G-17.2 which shows that depending on if the initial equilibrium was stable or not, we may have an increase or decrease in $k_{t+1}$. Particularly, if the initial point is stable, $k$ will fall and if unstable, then $k$ will rise. So if you do the above analysis of determining which way the $A$ function is going to move and you start at the SS, then you will need to know the stability properties of the initial equilibrium. See Diamond text.

- Usually, we are at a stable equilibrium and an increase in government debt (internal), will lower the capital stock. This crowds out private capital accumulation through equation (2). Savings will change because the net wages have fallen (eqn 1), and the burden of the debt is to reduce the capital of future generations. This is only a good thing if there is dynamic inefficiency and it is actually pareto improving to have less capital accumulation. Usually this is not the case.

# 18 Lecture 18: November 3, 2004

## 18.1 More on OLG

- Consider the utility effect of debt. Write utility as:

$$u(c^y, c^o) = u(c^y(\hat{w}, r), c^o(\hat{w}, r)) \equiv V(\hat{w}, r),$$

Where $\hat{w}$ and $r$ are the factor prices. Thus the differential wrt $b$:

$$\frac{du}{db} = \frac{\partial V}{\partial \hat{w}} * \frac{\partial \hat{w}}{\partial b} + \frac{\partial V}{\partial r} * \frac{\partial r}{\partial b}.$$

Where:

$$\frac{\partial V}{\partial \hat{w}} = \frac{\partial u}{\partial c^y} * \frac{\partial c^y}{\partial \hat{w}} + \frac{\partial u}{\partial c^o} * \frac{\partial c^o}{\partial \hat{w}}.$$
$$\frac{\partial V}{\partial r} = \frac{\partial u}{\partial c^y} * \frac{\partial c^y}{\partial r} + \frac{\partial u}{\partial c^o} * \frac{\partial c^o}{\partial r}.$$

Thus, (****DERIVE****)

$$\frac{du}{db} = -\frac{\partial u}{\partial c^y}[(r-n) + b\frac{dr}{db} + (k - \frac{s}{1+r})\frac{dr}{db}].$$

Recall that $\hat{w} = w - (r-n)b$. Then if all the debt is internal: (****DERIVE****)

$$\frac{du}{db} = \underbrace{-\frac{\partial u}{\partial c^y}}_{-}(r-n)\underbrace{[1 + \frac{k+b}{1+r}\frac{dr}{db}]}_{+}.$$

So the crucial term in all of this is $r - n$.

  - If $r > n \Longrightarrow k^{ss} < k^{GR} \Longrightarrow \frac{\partial u}{\partial b} < 0$.
  - If $r < n \Longrightarrow k^{ss} > k^{GR} \Longrightarrow \frac{\partial u}{\partial b} > 0$. Too much capital.

## 18.2 Optimal Growth

- So what happens when we endogenize the savings rate (along with the factor prices)? Under what conditions will the market solution be optimal? With OLG, we had a finite horizon so usually the market solution was not optimal. But we never assumed that there was any cost to reaching the optimum. In the next section, we will assume both finite and infinite horizons and also the cost of getting to the optimum.

- Define the Salvage Value of Capital as the utility value, $S(k_T)$ of capital at a certain time period. It might be the goal value of capital at time $T$ and we ask the question of how do we reach that level?

- Assume $S(k_{T+1}) = 0$ which implies $k_{T+1} = 0$, ie there is no reason to hold capital beyond period $T$. Also assume $n = 0$, $L = 1$, and $\delta = 1$. Thus, we have the following technological constraint:
$$k_{t+1} + c_t = f(k_t). \quad (1)$$
The objective of the social planner is therefore:
$$Max_{\{c_t,k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t), \quad (2)$$
with $\beta \in (0, 1)$, $k_0$ given, and subject to (1).

- Assume that $u(c)$ is bounded which means the discounted flow of utility must be finite. Assume $u' > 0$, $u'' < 0$ and:
$$lim_{c_t \to 0} u'(c_t) = \infty.$$
This along with $f(0) = 0$ means that we will NOT have a corner solution so (1) binds. Substituting in, we rewrite the social planner's problem:
$$Max_{\{k_t\}_{t=1}^{T+1}} \sum_{t=0}^{T} \beta^t u(f(k_t) - k_{t+1}). \quad (3)$$
There might be an additional additive term: $\beta^{T+1} S(k_{T+1})$ but we have assumed this to be zero.

- Thus, the FOC $(k_t)$ is:
$$\beta u'(f(k_t) - k_{t+1}) f'(k_{t+1}) - u'(f(k_{t-1}) - k_t) = 0, \quad t = 1, \ldots, T.$$
Note there are 3 time periods of capital in this equation.

- Backwards Recursion. At $T$, (note $k_{T+1} = 0$), the FOC implies:
$$\beta u'(f(k_T)) f'(k_T) = u'(f(k_{T-1}) - k_T).$$
Note this only has 2 time periods of capital. Implicitly we can write:
$$k_T = g_{T-1}(k_{T-1}).$$
So a time $T$, we have some level of capital coming in, and we will behave optimally in that period no matter what we have to work with. In period $T - 1$, the FOC implies:
$$\beta u'(f(k_{T-1}) - g(k_{T-1})) f'(k_{T-1}) = u'(f(k_{T-2}) - k_{T-1}).$$
Or implicitly:
$$k_{T-1} = g_{T-2}(k_{T-2}).$$
Again, we'll do the best (optimal) with whatever we are given: "Today is the first day of the rest of your lives."

- Repeating this process we have the following policy functions:

$$k_T = g_{T-1}(k_{T-1}).$$

$$k_{T-1} = g_{T-2}(k_{T-2}).$$

$$\vdots$$

$$k_2 = g_1(k_1).$$

$$k_1 = g_0(k_0).$$

So the form of the solution is a sequence of decision rules. No matter where we start, we behave optimally from that period onwards. This is exactly the same as Bellman's Principal.

## 18.3 Infinite Horizon Case

- We use dynamic programming to solve in this case. First we must deal with the question of the utility function being bounded. If $u(c) = log(c)$, then clearly $u(c)$ is NOT bounded. However, if $f(0) = 0$ and the inada conditions hold for $f$, we know that $f$ has a maximum sustainable value $\overline{k} = f(\overline{k})$. This implies two solutions: $k = 0$ and $k = \overline{k}$. Thus, along any feasible growth path, it must be the case that:

$$k_t \in [0, \overline{k}].$$

If $k_t$ is bounded in this range, $c_t$ is also bounded which implies that $log(c_t)$ is bounded. Thus, we are ok with log utility.

- The Bellman's Equation will look like the following:

$$V(k_t) = Max_{0<y\leq\overline{k}}\left\{u(f(k) - y) + \beta V(k_{t+1})\right\}.$$

The maximum is attained at $y = g(k)$. However, we would need the form of $V$ to solve for this policy equation.

- Suppose we cannot solve for $V(k)$. What are the characteristics of $V(k)$? Well, via Stokey and Lucas, we know that if $u(\cdot)$ and $f(\cdot)$ are strictly increasing and concave, then $V(\cdot)$ is strictly increasing and concave with a continuous derivative, $V'(k)$, given by the envelope condition. This gives us a lot in terms of determining the optimal growth path. More next week.

# 19 Lecture 19: November 8, 2004

## 19.1 Optimal Growth - Infinite Horizon

- Recall our technological constraint:

$$f(k_t) + (1-\delta)k_t = c_t + (1+n)k_{t+1}.$$

Let $k_t = k$ and $k_{t+1} = k'$. Write the Bellman's Equation as follows:

$$V(k) = Max_{0<k'\leq \bar{k}}\left\{u(f(k) + (1-\delta)k - (1+n)k') + \beta V(k')\right\}.$$

- Steady State: Find $k = k' = g(k)$, our resulting policy function. First consider the FOC ($k'$):

$$u'(f(k) + (1-\delta)k - (1+n)k')(-(1+n)) + \beta V'(k') = 0.$$
$$(1+n)u'(f(k) + (1-\delta)k - (1+n)k') = \beta V'(k').$$
$$(1+n)u'(f(k) + (1-\delta)k - (1+n)g(k)) = \beta V'(g(k)).$$

And also the envelope condition:

$$V'(k) = u'(f(k) + (1-\delta)k - (1+n)k')[f'(k) + (1-\delta)].$$

$$V'(k) = u'(f(k) + (1-\delta)k - (1+n)g(k))[f'(k) + (1-\delta)].$$

- In the particular case of $n = 0$ and $\delta = 1$, we have:

$$u'(f(k) - g(k)) = \beta V'(g(k)).$$

$$V'(k) = u'(f(k) - k')[f'(k)].$$

Where the first equation is the trade off between consuming today and saving and consuming tomorrow while the second gives us the value of a unit of capital.

- Combine the FOC and envelope condition noting that $k = k' = g(k)$:

$$(1+n)u'(f(k)+(1-\delta)k-(1+n)g(k)) = \beta u'(f(k)+(1-\delta)k-(1+n)g(k))[f'(k)+(1-\delta)].$$

$$(1+n) = \beta[f'(k) + (1-\delta)].$$

Solve for $f'(k)$ to get a version of the Modified Golden Rule:

$$f'(k) = \frac{1+n}{\beta} - 1 + \delta.$$

- Recall if we just maximize consumption, $c = f(k) - (n+\delta)k$ yields the Golden Rule: $f'(k) = n + \delta$.

- In the special case of $n = 0$ and $\delta = 1$ we have:

$$\text{Golden Rule: } f'(k) = 1.$$

$$\text{Modified Golden Rule: } f'(k) = \frac{1}{\beta}.$$

The difference comes from the fact that the Golden Rule only considers the steady state optimum and not the cost of getting there. In the MGR, we consider the discount factor we place on consuming in the future. If $\beta < 1$, $\frac{1}{\beta} > 1$ and $k^{MGR} < k^{GR}$.

- To determine the dynamics of the solution we make use of two propositions proved in the notes:
  - (1) $g(k)$ is increasing in $k$.
  - (2) $g(k)$ is $\gtreqless k$ as $k \lesseqgtr k^{MGR}$

- See G-19.1. There must be exactly one crossing and it is stable.

- Finally we would like to compare the social planner's solution to the market solution to determine if it is optimal. When will the competitive equilibrium (CE) be optimal? Given the social planner's solution, what market prices will support it?

- Assume 2 types of agents: Households and Firms.
  - Households own all factors of production, buy goods from firms, consume and save.
  - Firms hire capital and labor to produce and sell output, and then return the profits to the household.

- Denote the following prices:
  - $p_t$ = time 0 price of a unit of output at time $t$ in some unit of account (say dollars).
  - $w_t$ = time $t$ price of labor in terms of time $t$ goods (real wage).
  - $r_t$ = real rental price of capital in terms of time $t$ output.

- Firm's Problem. Facing a price vector $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, with production function $F(k_t, n_t)$ where $n_t$ is labor input PER worker, the firm chooses a sequence $\{k_t, n_t\}_{t=0}^{\infty}$ to maximize:

$$\pi = \sum_{t=0}^{\infty} p_t[F(k_t, n_t) - w_t n_t - r_t k_t]. \quad (10).$$

- Household's Problem. Facing a price vector $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, the household chooses a sequence $\{c_t, k_{t+1}, n_t^s\}_{t=0}^{\infty}$, where $n_t^s$ is the supply of labor, to maximize:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \quad (11)$$

such that:

$$\sum_{t=0}^{\infty} p_t(c_t + k_{t+1}) \leq \sum_{t=0}^{\infty} p_t(w_t n_t + r_t k_t) + \pi. \quad (12)$$

- **Definition:** A Competitive Equilibrium (CE) is a feasible sequence, $\{c_t, k_{t+1}, n_t, p_t, r_t, w_t\}_{t=0}^{\infty}$ which solves the household's and firm's problems where $n_t = n_t^s = 1$.

- Result 1: If a sequence $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ is a CE then is solves the planner's growth problem. Proof: Suppose not. Suppose there exists a feasible sequence: $\{\hat{c}_t, \hat{k}_{t+1}\}_{t=0}^{\infty}$ yielding a higher value of (11). Thus (12) must be violated (and under a CE profits are 0), ie:

$$\sum_{t=0}^{\infty} p_t(\hat{c}_t + \hat{k}_{t+1}) > \sum_{t=0}^{\infty} p_t(w_t \hat{n}_t + r_t \hat{k}_t). \quad (*)$$

However the sequence is feasible so:

$$\hat{c}_t + \hat{k}_{t+1} \leq F(\hat{k}_t, \hat{n}_t) \quad (**).$$

Substituting equation (**) into (*)

$$\sum_{t=0}^{\infty} p_t(F(\hat{k}_t, \hat{n}_t)) > \sum_{t=0}^{\infty} p_t(w_t \hat{n}_t + r_t \hat{k}_t). \quad (*)$$

But then the feasible sequence must yield a higher value of (10) than $\{c_t, k_{t+1}\}$ and thus the sequence $\{c_t, k_{t+1}\}$ must not be optimal.

- Result 2: The optimal sequence $\{\tilde{c}_t, \tilde{k}_{t+1}\}$ can be supported as a CE with:
  - $\tilde{w}_t = f(\tilde{k}_t) - \tilde{k}_t f'(\tilde{k}_t) > 0$.
  - $\tilde{r}_t = f'(\tilde{k}_t)$.
  - $\tilde{p}_t = p_0 \beta^t \dfrac{u'(\tilde{c}_t)}{u'(\tilde{c}_0)}$.

  From Drazen Notes: "The basic idea here is that if there are no externalities, one can find a set of prices that support an optimal sequence. Another cheer for capitalism in that a price system can decentralize an optimal solution. That is, allow it to be achieved by a market system rather than it being necessary for a central authority to control quantities directly."

# 20 Lecture 20: November 10, 2004

## 20.1 Stochastic Growth Model

- Consider the technological constraint:

$$c_t + k_{t+1} \leq \mathfrak{z}_t f(k_t).$$

Where $\mathfrak{z}$ is a stochastic productivity shock. Thus $\{\mathfrak{z}\}$ is a sequence of iid random variables. A $\mathfrak{z}$ can take on values:

$$Z^1, Z^2, \ldots, Z^S,$$

with probabilities: $\theta^s$. This yields a sequence of realizations:

$$\mathfrak{z}_0 = Z(0) \in Z^S, \ \ \mathfrak{z}_1 = Z(1) \in Z^S, \ldots.$$

- The social planner maximizes:

$$E_0[\sum_{t=0}^{\infty} \beta^t u(c_t)] = \sum_{t=0}^{\infty} \beta^t E_0 u(c_t).$$

Note that the SP knows the realizations of $\mathfrak{z}_t$ at time $t$ and also the distribution of future $\mathfrak{z}$'s. Thus the choice of $k$ will be conditional on $\mathfrak{z}_t$.

- The Bellman's Equation is therefore:

$$V(k, \mathfrak{z}) = Max_{k'} \left\{ u(\mathfrak{z} f(k) - k') + \beta E V(k', \mathfrak{z}') \right\}.$$

Or writing out the expectation term:

$$V(k, \mathfrak{z}) = Max_{k'} \left\{ u(\mathfrak{z} f(k) - k') + \beta \sum_{s=1}^{S} \theta^s V(k', Z^s) \right\}.$$

Note that the distribution of $\mathfrak{z}$ could be iid (non-dynamic) or possibly markov.

- The policy function will be of the form:

$$k_{t+1} = g(k_t, \mathfrak{z}_t).$$

- So what does the time path of $k$ look like? It will depend on the realization of the $\mathfrak{z}$'s. We could just generate one sample time path and compare the path of $k$ to the actual economy but that would be prone to many errors. Instead, run a monte carlo simulation and then look at the entire distribution of the path of $k$ and compare the moments of $k$, $c$, etc to the actual economy. Method of Moments technique.

- Specific Example: $f(k) = k^\alpha$, $u(c) = log(c)$. This implies:

$$k_{t+1} = \alpha\beta\mathfrak{z}_t k_t^\alpha. \quad (*)$$

Denote the CDF of $\mathfrak{z}$ to be $M(\mathfrak{z})$.

- Thus the conditional CDF of $k_{t+1}$ is as follows:

$$\begin{aligned} H(k_{t+1}|k_t) &= H(K'|K) \\ &= Prob(k_{t+1} \leq K'|k_t = K) \\ &= Prob(\alpha\beta\mathfrak{z}_t K^\alpha \leq K') \\ &= Prob(\mathfrak{z}_t \leq \frac{K'}{\alpha\beta K^\alpha}) \\ &= M(\frac{K'}{\alpha\beta K^\alpha}) \end{aligned}$$

- Denote $H^t(k_t)$ the CDF of $k_t$. The associated PDF is therefore:

$$\frac{dH^t(k_t)}{dk_t} = h^t(k_t) = Prob(k_t = K).$$

- Thus the CDF of $k_{t+1}$ is:

$$\begin{aligned} H^{t+1}(k_{t+1}) &= \int_{k_t} M(\frac{k_{t+1}}{\alpha\beta k_t^\alpha}) h^t(k_t) dk_t \\ &= \int_{k_t} M(\frac{k_{t+1}}{\alpha\beta k_t^\alpha}) dH^t(k_t) \end{aligned}$$

Where the second equation comes from substituting from the definition of the PDF.

- The steady state in this situation is going to be a distribution instead of a particular value. (Ergodic Distribution). Thus the steady state will be:

$$H(k_{t+1}) = \int_{k_t} M(\frac{k_{t+1}}{\alpha\beta k_t^\alpha}) dH(k_t).$$

This equation would characterise the steady state. We could generate from this and compare moments. Issues with this include data-mining, the distribution of $\mathfrak{z}$ and the assumption that the real world is also in a steady state and not a transition state.

## 20.2 Endogeneous Growth Models

- Technological progress is key to growth in the economy and any good model should include it. Many of the solow assumptions simply are not true so we need a better model.

- Consider a labor-augmenting production function:

$$Y_t = F(K_t, L_t, t) = F(K_t, A_t L_t).$$

Define:

$$\tilde{k}_t = \frac{K_t}{A_t L_t},$$
$$\tilde{y}_t = \frac{Y_t}{A_t L_t}.$$

- This yields a dynamic equation for $k$:

$$\dot{\tilde{k}} = sf(\tilde{k}) - (n + \delta)\tilde{k}.$$

- In the steady state, $\tilde{k}$ is constant but $k$, $w$, and $y$ all grow at rate:

$$a = \frac{\dot{A}}{A}.$$

- But still, where does this "$A$" come from? New models will endogenize $A$. If all countries had the same access to $A$, they should all grow at the same rate, which is not true.

- Two Questions:
  - (1) Why do countries get stuck with low or negative growth ?
  - (2) Why do some countries start to take off? What causes the transition to high growth?

# 21 Lecture 21: November 15, 2004

## 21.1 Romer's $R\&D$ Model

- Suppose $n = \delta = 0$. Assume some fraction, $a_L$, of labor is involved in $R$ and $D$. Thus,

  $$Y_t = K_t^{\alpha}[(1 - a_L)LA_t]^{1-\alpha}.$$

  Where $L$ is constant and $LA_t$ is effective labor. Assume:

  $$\dot{A} = Ba_L LA_t,$$

  Or,

  $$\frac{\dot{A}}{A} = BA_L L.$$

  So technological change is now endogeneous to the model.

- So there is a trade off between higher growth in the future and the current level of growth today as we change $a_L$.

- The result is that if a country wants to grow more quickly, it simply has to devote more labor to $R$ and $D$. This labor could be UNSKILLED! We have not made any assumptions about how efficiently a country can adopt the technology. Therefore any country should be able to "catch up."

## 21.2 Learning by Doing

- Suppose production is now:

  $$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}.$$

  Now suppose that:

  $$A_t = BK_t^{\phi},$$

  so the more we devote to capital, the more technology will grow and thus the more efficient is the labor force. Ie, if we use the machines a lot, we get better at using them.

- Substituting in:

  $$Y_t = K_t^{\alpha} B^{1-\alpha} K_t^{\phi(1-\alpha)} L_t^{1-\alpha}.$$

  $$Y_t = K_t^{\phi(1-\alpha)+\alpha} B^{1-\alpha} L_t^{1-\alpha}.$$

- Accumulation of physical capital equation:

  $$\dot{K}_t = sY_t - \delta K_t.$$

  If $\delta = 0$,

  $$\dot{K}_t = sY_t = sK_t^{\phi(1-\alpha)+\alpha} B^{1-\alpha} L_t^{1-\alpha}.$$

- Technological change. Recall $A_t = BK_t^{\phi}$, so

$$\frac{dA_t}{dt} = \dot{A}_t = B\phi K_t^{\phi-1}\dot{K}_t.$$

- If $\phi = 1$,

$$Y_t = K_t B^{1-\alpha} L_t^{1-\alpha}.$$
$$Y_t = (BL_t)^{1-\alpha} K_t.$$

  Sometimes called the "$AK$" model. Also,

$$\frac{\dot{K}_t}{K} = \frac{sK_t^{1(1-\alpha)+\alpha} B^{1-\alpha} L_t^{1-\alpha}}{K}.$$

$$\frac{\dot{K}_t}{K} = s(BL_t)^{1-\alpha}.$$

  And because of the relationship between $A$ and $K$,

$$\frac{\dot{A}}{A} = \frac{\dot{K}}{K} = s(BL_t)^{1-\alpha}.$$

  So countries with a higher population growth will have higher technological growth, which is hard to believe. Note that output is a linear function of capital under these assumptions which means that invoking technological progress in the model has eliminated the decreasing returns to scale for capital.

- Again, as in the last model, we are ignoring the skill factor of labor, so there is more to be desired. We also ignore the political implications and the willingness to adapt new technologies across countries.

## 21.3 Azariadis/Drazen Human Capital Model

- Why do some countries get stuck? Egypt had an average of 4.8% growth per decade over the first half of the 20th century while in England, it was more like 20%.
- In this model, we look for multiple equilibria to explain why country's growth rates do not converge (if not their levels).
- As in the Diamond model where individuals made decisions about saving without taking into account their effect on future generations (an externality), here agents may select a certain amount of training in period 1 in order to make them more productive (yielding a higher effective wage) in the second period.
- So we have a two period model where the agents work in both periods and everyone has a time endowment of 1 in each period. Mr. $i$ has efficiency units $x_t^i$ in the first period and $x_{t+1}^i$ in the second. Suppose $x_t^i$ is given and:

$$x_{t+1}^i = x_t^i * h(\tau_t, \overline{x}_t).$$

Where $\tau_t$ is the time spent training in period $t$, $\tau_t \in (0,1)$, and $\overline{x}_t$ is the average amount of human capital that is available to all. So the guy comes in with some level of education and maximizes his lifetime wealth by choosing how much training to receive now which in turn effects his income tomorrrow. The degree to which he can increase his efficiency units tomorrow also depends on how much knowledge is available around him.

- So the agent choose $\tau_t$ to maximize:

$$I = \overbrace{(1-\tau_t^i)}^{Time\ Working\ in\ t} \underbrace{w_t x_t^i}_{Effective\ Wage} + \underbrace{\frac{w_{t+1}x_{t+1}^i}{R_{t+1}}}_{Tomorrow's\ Discounted\ Income}.$$

Note the agent doesn't do any more training in the second period, he's just left with whatever knowledge he's accumulated, $x_{t+1}^i$.

- Rewrite the objective function:

$$I = (1-\tau_t^i)w_t x_t^i + \frac{w_{t+1}x_{t+1}^i}{R_{t+1}}.$$

$$I = (1-\tau_t^i)w_t x_t^i + \frac{w_{t+1}x_t^i h(\tau_t, x)}{R_{t+1}}.$$

- FOC ($\tau_t$):

$$-w_t x_t + \frac{w_{t+1}x_t}{R_{t+1}} h_\tau(\tau, x) \leq 0.$$

$$-w_t + \frac{w_{t+1}}{R_{t+1}} h_\tau(\tau, x) \leq 0.$$

$$w_t \geq \frac{w_{t+1}}{R_{t+1}} h_\tau(\tau, x).$$

$$R_{t+1} \geq \frac{w_{t+1}}{w_t} h_\tau(\tau, x). \quad (**)$$

- So the LHS is the return on capital and the RHS is the return on human capital. If we had an equality, this would make intuitive sense because we have a tradeoff between two assets, capital and human capital. In equilibrium, their returns should be the same or else we would reallocate.

- If the relation was a strict $>$, no agent would take any training and $\tau = 0$. So depending on if the equality binds, ($\tau > 0$), or if the equality does not bind ($\tau = 0$), we will get different equilibria.

- Consider a reference case: $A_t(\cdot) = A_t$ and $h(\tau, \overline{x}) = h(\tau)$, so there are NO externalities. Then equation (**) would imply one unique equilibrium and we haven't explained anything. We need some sort of externality to get the multiple equilibria. Let's assume there is an externality in the training function so $h$ is a function of both personal training and economy-wide knowledge.

- Lets assume the $h$ function takes on a specific form:

$$h(\tau, \overline{x}) = 1 + \gamma(\overline{x})\tau^i, \quad \gamma' > 0, \quad lim_{x\to\infty}\gamma(\overline{x}) = \hat{\gamma}.$$

  Thus,

$$x^i_{t+1} = x^i_t * h(\tau_t, \overline{x}_t) = x^i_t(1 + \gamma(\overline{x})\tau^i) = x^i_t + \gamma(\overline{x})\tau^i x^i_t.$$

  This seems like a fairly unrestrictive assumption but we are assuming that knowledge is completely domestic and that education is free. Note that $h_\tau(\tau, x) = \gamma(\overline{x})$, so equation (**) becomes:

$$R_{t+1} \geq \frac{w_{t+1}}{w_t}\gamma(\overline{x}).$$

- Suppose we assume there is some $x_0$ which implies $\gamma(x_0) = \gamma_0$. So we have an economy where the representative agent starts out with knowledge level $x_0$ and this implies a return on training of $\gamma_0$. Assume our FOC holds with equality yields:

$$R_{t+1} = \frac{w_{t+1}}{w_t}\gamma_0. \quad (*)$$

  Now equation (*) gives us the return on capital overtime for a given level of initial knowledge (and given wages rates). This, via the production function (??) defines a capital accumulation frontier (G-21.1). It's our usual upward sloping concave function mapping current values of capital into tomorrow's values of capital.

  - The idea is that if we start out at some point below the line (where $k_t$ is low compared to $k_{t+1}$), the return on using more capital is relatively large so equation (*) becomes:

$$R_{t+1} > \frac{w_{t+1}}{w_t}\gamma_0.$$

    The return on capital is TOO BIG and $\tau = 0$, ie there is no training. The agents decide to devote all resources to capital accumulation instead of human capital accumulation. In this case, consider the evolution of $x_t$. Recall:

$$x^i_{t+1} = x^i_t + \gamma(\overline{x})\tau^i x^i_t.$$

    So, if $\tau^i = 0$,

$$\frac{x^i_{t+1}}{x^i_t} = 1 + \gamma(\overline{x})\tau^i = 1.$$

    Or

$$x^i_t = x^i_{t+1},$$

    so we don't get any more efficient in the second period. $x$ is constant in the "no-training" equilibrium.

– Now consider starting above the line in G-21.1. The return on capital is relatively low so for

$$R_{t+1} = \frac{w_{t+1}}{w_t}\gamma_0,$$

it must be the case that $\tau > 0$. Ie, given a starting point, $k_0$, $k_1$, we have:

$$R_1 = \frac{w_1}{w_0}\gamma_0,$$

so the initial level of $x_0$ must be high enough to make this equality hold and allow for $\tau > 0$. In this situation, our evolution of $x_t$ becomes:

$$\frac{x^i_{t+1}}{x^i_t} = 1 + \gamma(\overline{x})\tau^i.$$

$$\frac{x^i_1}{x^i_0} = 1 + \gamma(x_0)\tau_0 > 0.$$

Or,

$$x^i_{t+1} > x^i_t,$$

so $x$ grows in the "training" equililbrium, and

$$\gamma_1 > \gamma_0.$$

- Thus there exists some threshold level of $x_0$ such that $\tau > 0$, otherwise, there will be no training and we end up in the old equilibrium. So we will get two steady states depending on the initial level of $x_0$. This may explain why some countries get stuck while others grow at a much more rapid rate.

# 22 Lecture 22: November 17, 2004

## 22.1 Time Inconsistency

- Consider a policy that is to take effect at time $t+s$ but which is decided upon either at time $t$ or at time $t+s$. If the policies are time consistent (TC):

$$\tau_{t+s}(t+s) = \tau_{t+s}(t).$$

  Often times, if there is a TC problem, this will not be the case.

- Recall with Shea, we found Time InConsistency (TIC) using hyperbolic discounting. We will not assume that here but will show situations where it still may arise.

**Example 1: Exam Date Announcement**

- Students are given a date for a final exam at some point in the future. After spending weeks studying the optimal policy on exam day is to cancel the exam – it would be a waste of everyone's time to take it since clearly the students have invested in human capital and know the material. If the game is repeated, then the prof would need to develop some sort of reputation or else the threat to have an exam would not be credible and students wouldn't study.

**Example 2: The Positive Theory of Inflation - Notes from the LSE**

- Consider the following model. First an expectations augmented aggregate supply curve:

$$y_t = \alpha(p_t - E_{t-1}[p_t]) = \alpha(\pi_t - E_{t-1}[\pi_t]).$$

  Where $\pi_t = p_t - p_{t-1}$.

- And consider a loss function for the economy as follows:

$$L_t = \pi_t^2 + \lambda(y_t - y^*)^2.$$

  Where $\lambda$ is the relative weight put on deviations from the long run equilibrium level (growth rate) of output. Since societies dislike instability, deviations, positive or negative, are viewed as bad for the economy.

- So there are two ways to model the ways in which policy makers make decisions: Commitment or Discretion.

- Commitment:

  - Set $\pi_t$ before $E_{t-1}[\pi_t]$ is developed. Assume the central bank has perfect control over the rate of inflation.

– Thus, clearly, to minimize the loss function, $L$, the central bank should make sure $\pi_t^c = 0$, or the growth rate of inflation is zero. By doing this however, they have committed themselves to this strategy so people's expections, being rational, will expect just this, so $E_{t-1}[\pi_t] = 0$. Thus the aggregate supply function becomes:

$$y_t = \alpha(\pi_t - E_{t-1}[\pi_t]) = \alpha(0 - 0) = 0.$$

Noting this is the growth rate of output so we are just saying that output will not grow under these circumstances. Thus our loss function:

$$L_t = \pi_t^2 + \lambda(y_t - y^*)^2 = 0 + \lambda(0 - y^*)^2 = \lambda y^{*2}.$$

$$L_t^c = \lambda y^{*2}.$$

- Discretion:

  – Here, we set $\pi_t$ optimally after $E_{t-1}[\pi_t]$ is formed. So we wait to figure out what people are going to expect inflation to be, and then we set policy instruments to determine the actual level of inflation such that it minimizes $L$. Note how the commitment strategy is not a Nash equilibrium because the central bank will have the incentive to renege on its strategy. Here we seek to minimize the loss function:

$$min_{\pi_t} L_t = \pi_t^2 + \lambda \left[ \alpha(\pi_t - E_{t-1}[\pi_t]) - y^* \right]^2.$$

  – FOC:

$$2\pi_t + 2\lambda\alpha \left[ \alpha(\pi_t - E_{t-1}[\pi_t]) - y^* \right] = 0.$$

$$\pi_t + \lambda\alpha \left[ \alpha(\pi_t - E_{t-1}[\pi_t]) - y^* \right] = 0.$$

$$\pi_t + \lambda\alpha^2\pi_t - \lambda\alpha^2 E_{t-1}[\pi_t] - \alpha\lambda y^* = 0.$$

$$(1 + \lambda\alpha^2)\pi_t = \lambda\alpha^2 E_{t-1}[\pi_t] + \alpha\lambda y^*.$$

  – In this setting, we have been taking $E_{t-1}[\pi_t]$ as GIVEN. But now look at the optimisation. Suppose we were following the commitment regime and therefore, $E_{t-1}[\pi_t] = 0$. Then $\pi_t = \dfrac{\alpha\lambda y^*}{1 + \lambda\alpha^2} \neq E_{t-1}[\pi_t]$. Thus the mere announcement of intentions to have zero inflation is NOT credible. So the only credible announcement is one in which $E_{t-1}[\pi_t] = \pi_t$. Thus our FOC becomes:

$$(1 + \lambda\alpha^2)\pi_t = \lambda\alpha^2\pi_t + \alpha\lambda y^*.$$

$$\pi_t^d = \alpha\lambda y^*.$$

And the loss function becomes:

$$L_t = \pi_t^2 + \lambda\Bigg[\alpha(\pi_t - E_{t-1}[\pi_t]) - y^*\Bigg]^2 = (\alpha\lambda y^*)^2 + \lambda y^{*2}.$$

$$L_t^d = \lambda y^{*2}(\alpha^2\lambda + 1).$$

- Now comparing the loss functions under commitment and discretion:

$$L_t^d - L_t^c = \lambda y^{*2}(\alpha^2\lambda + 1) - \lambda y^{*2} = \lambda y^{*2} + \lambda^2\alpha^2 y^{*2} - \lambda y^{*2} = \lambda^2\alpha^2 y^{*2} = (\lambda\alpha y^*)^2.$$

- This extra term is the deadweight loss associated with the government's inability to commit to zero inflation ex-ante.
- So what all this says is that when we impose a restriction on the model, such as the commitment approach, the result is actually less efficient which is counter-intuitive. The reasoning behind this is TIC.
- Ways to get around TIC include reputation effects that come into play say when governments are in power for a longer period of time and care about their reputations. It is also possible to impose a constitutional rule forbidding reneging at a later date. This however, puts restraints on policy stabilization when shocks hit the economy.

**Example 3: Debt Determination**

- A country that issues debt in its domestic currency could simply inflate their currency thereby reducing the value of the debt that must be repaid. Countries could promise not to do this but its not credible. Hence, often debt is issued in the buyer's currency instead as a sort of insurance policy.

**Model of Capital Taxation**

- 2 period model with technological and output constraints:

$$c_1 + k = y, \quad c_2 + g = an + Rk.$$

Where $y$ is an endowment in period 1 which consumers divide between consumption and capital savings. In period 2, we have production from labor, $an$, as well as the return on the capital savings from period 1, $Rk$. In addition to consumption spending in period 2, we also use the resources to finance government expenditure, $g$. Denote the utility of a representative agent as:

$$U_1 = ln(c_1) + \beta[ln(c_2) + \delta ln(1-n) + \gamma ln(g)].$$

Where $\beta$, $\delta$, and $\gamma$ are fixed coefficients, and $1-n$ is leisure time.

- The Command Solution (Social Planner's Solution - First Best) is as follows. First rewrite the problem as:

$$Max_{k,n,g} U_1 = ln(y-k) + \beta[ln(an + Rk - g) + \delta ln(1-n) + \gamma ln(g)].$$

FOCs:

$$\frac{\partial U}{\partial k} = 0 \Rightarrow \frac{1}{y-k} = \frac{\beta R}{an + Rk - g}. \quad (1)$$

$$\frac{\partial U}{\partial n} = 0 \Rightarrow \frac{\beta a}{an + Rk - g} = \frac{\delta}{1-n}. \quad (2)$$

$$\frac{\partial U}{\partial g} = 0 \Rightarrow \frac{\beta}{an + Rk - g} = \frac{\gamma}{g}. \quad (3)$$

Which can be simplified to:

$$c_1 = \frac{y + a/R}{1 + \beta(1 + \delta + \gamma)},$$

$$c_2 = \beta R c_1.$$

$$n = 1 - \frac{\delta}{a}\beta R c_1.$$

$$g = \gamma \beta R c_1.$$

$$k = \frac{\beta(1 + \delta + \gamma)y - a/R}{1 + \beta(1 + \delta + \gamma)}.$$

So for the command solution, simply set each control as above and we attain the first best solution.

- Now suppose the social planner can only use distortionary taxation and cannot allocate quantities directly. With this policy, there are now substitution effects in addition to income effects of the taxation. Knowing there will be a tax on capital will induce the first period consumer to save less. So the individual budget constraint becomes:

$$c_1 = y - k, \quad c_2 = (1-\tau)Rk + (1-\mu)an.$$

So the government's budget is of course:

$$g = \tau Rk + \mu an.$$

The crucial point here is that the "Representative Agent" takes $g$ as GIVEN when choosing optimal $c_1$ and $k$. This is because he is one of many small (atomistic) consumers so his decision will have no effect on government expenditures. Taken as a whole, of course all agents decisions will effect the governments choice of $\tau$ and $\mu$, but since the representative agent does not internalize the government's budget constraint, he just takes $g$ as given. So in period 1, consumers make their choices based on $\tau^e$ and

$\mu^e$, their expectation of future tax rates. Hence:

$$c_1 = c(\tau^e, \mu^e), \quad k = k(\tau^e, \mu^e).$$

In period two:

$$c_2 = c(\tau, \mu, \tau^e, \mu^e), \quad n = n(\tau, \mu, \tau^e, \mu^e).$$

The government's budget constraint becomes:

$$g = \tau R k(\tau^e, \mu^e) + \mu a n(\tau, \mu, \tau^e, \mu^e).$$

Lets suppose that the representative agent expects the tax rates to be $\tau^a$ and $\mu^a$ which are the announcement values that the government makes in the first period. The government then maximizes in period 2 the indirect utility function:

$$V(\tau, \mu, \tau^a, \mu^a).$$

Will $\tau = \tau^a$ ? Not here (also not shown!). In general $\tau \neq \tau^a$ so we have a TIC solution. The optimal policy for the government is to announce a low $\tau$ in period 1 to induce people to save and then raise $\tau$ (to 1?) so they tax it all away. Fooling people by this method is better (for everyone - including the consumers) than just setting the tax rates equal to people's expectations. This leads to the next option for the government.

- Suppose the government can precommit to a policy so that it has to set next period tax rates equal to whatever they announce this period. The mechanism for this to be viable could be reputation effects if the game is repeated.

- Finally, we could analyze the time consistent solution (TC). Suppose there is no mechanism to precommitment. As of $t = 2$, the government preform's the following maximization:

$$Max_{\tau,\mu} U_2 = ln(c_2) + \delta ln(1-n) + \gamma ln(\underbrace{\tau R k + \mu a n}_{g}), \quad k \ given.$$

  The solution will be of the form:

$$\tau = \tau(k), \mu = \mu(k),$$

  reaction functions. In period 1, the consumers perform the following maximization:

$$Max_k\, U_2 = ln(c_2) + \delta ln(1-n) + \gamma ln(g), \quad \tau \ given.$$

  The solution will be of the form:

$$k = k(\tau),$$

  another reaction function. So the intersection of $\tau(k)$ and $k(\tau)$ will be the TC solution.

- In general, we have the following rankings for utility from the various tax-setting

schemes:
$$U^{Command} \geq U^{TIC} \geq U^{Commitment} \geq U^{TC}.$$

Note this means that the government is actually better off by tying it's hands then by finding the TC solution. Since the commitment strategy is one possible TIC strategy, the utility from TIC must be at least as big as commitment. Finally, the command solution is always first best.

- So why do people need to fooled in order to attain a good solution to this problem? Is it just the dynamic nature of the decisions? Note quite. The important point is that having a single agent versus a representative agent causes a conflict of interest. We don't choose our actions all together but rather just maximize our own utility. The macro effects of this cause a TIC.

- Consider the "Single Agent Problem" where the agent now internalizes the government's budget constraint. We can even switch around the timing in two ways:

  - (1) Suppose the individual chooses $k$ and then the government chooses $\tau$. In period 2, the government solves:

    $$Max_\tau U(\tau, k), \quad k \; given \Longrightarrow \tau = \tau(k).$$

    In period 2, the agent solves:

    $$Max_k U(\tau(k), k), \quad \Longrightarrow k^*.$$

    Yielding:
    $$u'(c_1) = \beta R u'(c_2), \quad u'(c_2) = v'(g),$$
    the solution to the Command Problem.

  - (2) Suppose the government chooses $\tau$ and then the individual chooses $k$. In period 2, the agent solves:

    $$Max_k U(\tau, k), \quad \tau \; given \Longrightarrow k = k(\tau).$$

    In period 2, the government solves:

    $$Max_\tau U(\tau, k(\tau)), \quad \Longrightarrow \tau^*.$$

    Yielding:
    $$u'(c_1) = \beta R u'(c_2), \quad u'(c_2) = v'(g),$$
    the solution to the Command Problem.

# 23 Lecture 23: November 22, 2004

## 23.1 One Factor Problem - Representative Agent

- So again turn to the one factor problem but now consider a representative agent facing a government who is using distortionary taxation. Constraints are:

$$c_1 = y_1 - k.$$

$$c_2 = y_2 + (1-\tau)Rk.$$

$$g = \tau Rk.$$

Utility for the rep agent is therefore:

$$u(\tau, k) = u(y_1 - k) + \beta u(y_2 + (1-\tau)Rk) + \beta v(g).$$

Note the agent takes the government's expenditure as given.

- Again, we consider two different orderings of events where with the individual or the government goes first.
- Case 1: Individual chooses $k$ and then the government chooses $\tau$.
  - Government maximizes:

$$Max_\tau \; u(\tau, k), \quad \overline{k} \; given.$$

The government is acting second and takes the level of capital (a NUMBER) as given. The FOC is therefore:

$$u'(y_2 + (1-\tau)R\overline{k}) = v'(\tau R\overline{k}). \quad (16)$$

So we have the marginal utility of consumption equal to the marginal utility of government expenditure.
  - Individual maximizes:

$$Max_k \; u(y_1 - k) + \beta u(y_2 + (1-\tau)Rk) + \beta v(g).$$

FOC:

$$u'(y_1 - k) = \beta(1-\tau)Ru'(y_2 + (1-\tau)Rk). \quad (17)$$

So we have an euler equation where $\beta(1-\tau)R$ is the discounted after tax return on capital.

- Case 2: Government chooses $\tau$ and then the individual(s) chooses $k$. (Assume precommitment mechanism.)
  - Individual maximizes:

$$Max_k \; u(y_1 - k) + \beta u(y_2 + (1-\tau)Rk) + \beta v(\tau R\overline{k}).$$

The individual is acting second and does not maximize over $v(\cdot)$ since this term is not internalized. The FOC is therefore:

$$u'(y_1 - k) = \beta(1-\tau)Ru'(y_2 + (1-\tau)Rk). \quad (17)$$

(17) again! Solving this would yield capital as a function of $\tau$, a REACTION function:

$$k = J(\tau).$$

– Government maximizes:

$$Max_\tau \; u(y_1 - J(\tau)) + \beta u(y_2 + (1-\tau)RJ(\tau)) + \beta v(\tau RJ(\tau)).$$

FOC:

$$-u'(c_1)J'(\tau) + \beta u'(c_2)(R(1-\tau)J'(\tau) - RJ(\tau)) + \beta v'(g)(RJ(\tau) + \tau RJ'(\tau)) = 0.$$

$$J'(\tau)[\underbrace{-u'(c_1) + \beta R(1-\tau)u'(c_2)}_{(17)=0}] - \beta u'(c_2)RJ(\tau) + \beta v'(g)(RJ(\tau) + \tau RJ'(\tau)) = 0.$$

$$-\beta u'(c_2)RJ(\tau) + \beta v'(g)RJ(\tau) + \beta v'(g)\tau RJ'(\tau) = 0.$$

$$\beta u'(c_2)RJ(\tau) = \beta v'(g)RJ(\tau) + \beta v'(g)\tau RJ'(\tau).$$

$$u'(c_2)J(\tau) = v'(g)J(\tau) + \tau v'(g)J'(\tau).$$

$$u'(c_2) - v'(g) = v'(g) \underbrace{\frac{\tau J'(\tau)}{J(\tau)}}_{Elasticity\ of\ \tau}. \quad (18)$$

- Comparing (18) to (16), we see that the two equations are different and therefore imply a different $\tau$. So we have time inconsistency coming out of the representative agent problem. Though having a sequential ordering of decisions need not always lead to TIC (as in the single agent problem) it clearly does here (rep agent problem).
- Since the agent is atomistic, he wants a low tax rate yet high government expenditures. So a high tax on everyone else while a low tax on his own capital is ideal. This creates the conflict of interest.

## 23.2 Richardian Equivalence

- Assume a NON-distortionary tax so we don't have to worry about TIC. Assume there is a representative agent and the government's budget constraint is:

$$b_0 + \sum_{t=0}^{T<\infty}(1+r)^{-t}g_t = \sum_{t=0}^{T<\infty}(1+r)^{-t}\tau_t.$$

Where $b_0$ is in the initial level of government debt and $\tau_t$ is the lump sum tax waged at time $t$.

- This intertemporal budget contraint of the goverment is known to all and is completely consistent over time.
- Suppose the individuals lives $k$ periods and faces lumps: $\tau_0, \tau_1, \ldots, \tau_k$.
- Suppose the government lives $T \geq k$ periods.
- Consider an example: $b_0 = 0$, $k = 2$, and $T = 2$. Government's IBC:

$$g_0 + \frac{g_1}{R} + \frac{g_2}{R^2} = \tau_0 + \frac{\tau_1}{R} + \frac{\tau_2}{R^2}.$$

- Utility of the agent is therefore:

$$\sum_{t=0}^{2} \beta^t u(c_t) + v(g_t).$$

- Question: Taking $\{g_t\}_{t=0}^{2}$ as given, is utility affected by changing the time path of taxes, $\{\tau_t\}_{t=0}^{2}$, while still satisfying the IBC? Ie, if we keep the constraint satisfied at all times but just rearrange when taxes are levied or when expenditures are made, is the agent made better or worse off? Answer: with $k = T$, there is absolutely no effect since the agent's euler will involve the PDV of $\tau$ and $g$ so changing the allocation over time, will not affect the agent's utility. RICHARDIAN EQUIVALENCE.
- 4 footnotes:
  - (1) Tax versus debt financing does not matter if the PDV of $g_t$ is unchanged.
  - (2) Value of the PDV matters as does the time path.
  - (3) If some of the taxes to finance $\sum_{t=0}^{k}(1+r)^{-t}g(t)$ come after period $k$ (the lifetime of the agent), then there is an affect on utility since the agent is reaping rewards for policies that he hasn't paid for. This is the $T > k$ case where Richardian Equivalence breaks down.
  - (4) Richardian Equivalence does NOT hold for distortionary taxation.

# 24 Lecture 24: November 24, 2004

## 24.1 Continuous time Optimization

- Consider the following continuous time optimization problem:

$$Max_{\vec{z}(t)} \int_{t=0}^{T} u(\vec{x}(t), \vec{z}(t))dt + S(\vec{x}_t), \quad (1a)$$

subject to:

$$\dot{x}_i(t) = G_i(\vec{x}(t), \vec{z}(t)),\ i = 1 \dots s, \quad (1b)$$

$$\vec{x}_i(0) = \vec{x}_{i0},\ i = 1 \dots s. \quad (1c)$$

So we choose $\vec{z}(t)$ to maximize our objective function which depends on the $s$ state variables, $\vec{x}$, as well as $s$ initial conditions on each state.

- Example of consumer's problem in continuous time:

$$Max_{c_t} \int_{t=0}^{T} e^{-\rho t} u(c_t)dt,$$

subject to:

$$\dot{k}(t) = f(k) - \delta k_t - c_t,$$

$$k_0 \ given.$$

- We solve these types of problems using a hamiltonian which we will outline below. Consider the following value function:

$$V(\vec{x}(t_0), t_0) = Max_{\vec{z}(t), t_0 \leq t \leq t_0+h} \left\{ \int_{t=t_0}^{t_0+h} u(\vec{x}(t), \vec{z}(t), t)dt + V(\vec{x}(t_0+h), t_0+h) \right\}.$$

We would like to simplify this bellman a bit by making the following three substitutions.

  - (1) Notice that for small enough $h$,

$$u(\vec{x}(t), \vec{z}(t), t) \approxeq u(\vec{x}_0, \vec{z}_0, t_0)\ \forall t_0 \leq t \leq t_0 + h.$$

  So the utility in the range of $t$ to $t+h$ for small enough $h$ is about constant.

  - (2) Take a first order taylor series approximation of the RHS value function:

$$V(\vec{x}(t_0+h), t_0+h) \approxeq V(\vec{x}(t_0), t_0) + \sum_{i=1}^{s} V_{x_i}(\vec{x}(t_0), t_0)(x_i(t_0+h) - x_i(t_0)) + V_t * h.$$

  Where here we have to differentiate $V$ wrt each state variable as well as time (the last term). Define:

$$q_i = V_{x_i} = \frac{\partial V(\vec{x}, t)}{\partial x_i},\ i = 1, \dots, s.$$

So the $q_i$'s are the shadow prices of each state variable and in a continuous setting, we call them the "co-state variables."

– (3) Note that:

$$x_i(t_0 + h) - x_i(t_0) = G_i(\vec{x}(t_0), \vec{z}(t_0)) * h.$$

- Substitute these 3 equations into our Bellman:

$$V(\vec{x}(t_0), t_0) = Max_{\vec{z}(t), t_0 \leq t \leq t_0+h} \left\{ u(\vec{x}(t_0), \vec{z}(t_0), t_0) * h + V(\vec{x_0}(t), t_0) + \sum_{i=1}^{s} q_i G_i(\vec{x}_0, \vec{z}_0) * h + V_t * h \right\}.$$

Pull out the $h$:

$$V(\vec{x_0}(t), t_0) = h * [Max_{\vec{z}(t)} \left\{ u(\vec{x}(t_0), \vec{z}(t_0), t_0) + \sum_{i=1}^{s} q_i G_i(\vec{x}_0, \vec{z}_0) \right\}] + V(\vec{x}_0(t), t_0) + V_t * h.$$

Simplify:

$$0 = h * [Max_{\vec{z}(t)} \left\{ u(\vec{x}(t_0), \vec{z}(t_0), t_0) + \sum_{i=1}^{s} q_i G_i(\vec{x}_0, \vec{z}_0) \right\}] + V_t * h.$$

$$-V_t * h = h * [Max_{\vec{z}(t)} \left\{ u(\vec{x}(t_0), \vec{z}(t_0), t_0) + \sum_{i=1}^{s} q_i G_i(\vec{x}_0, \vec{z}_0) \right\}].$$

$$-V_t = Max_{\vec{z}(t)} \left\{ u(\vec{x}(t_0), \vec{z}(t_0), t_0) + \sum_{i=1}^{s} q_i G_i(\vec{x}_0, \vec{z}_0) \right\}.$$

$$-V_t = Max_{\vec{z}(t)} H \equiv H^*.$$

Where the Hamiltonian is defined as:

$$H(\vec{x}, \vec{z}, \vec{q}, t) = u(\vec{x}, \vec{z}, t) + \sum_{i=1}^{s} q_i G_i(\vec{x}, \vec{z}).$$

And:

$$H^* = H(\vec{x}, \vec{z}^*, \vec{q}, t),$$

where $\vec{z}^*$ is the optimal choice of the control policy.

**Optimal Dynamic Paths** $(x, z, q)$

- We are given the dynamic equation for the states, but what about $z$ and $q$? For $z$, it's easy:

$$\frac{\partial H}{\partial z_k} = 0, \ k = 1, \ldots, n.$$

The hard part of $q$. To determine this, we have 4 steps:

– (1) Differeniate $H^*(\cdot)$ wrt each $x_i$ holding all else constant:

$$H^*_{x_i} = H_{x_i} + \underbrace{\sum_{k=1}^{n} H_{z_k} \frac{\partial z^*_k}{\partial x_i}}_{=0 \; by \; envelope} = H_{x_i}.$$

– (2) $H^*_{q_i} = H_{q_i}$ by the same reasoning. Since $H$ is linear in the shadow prices we have:

$$H^*_{q_i} = \frac{\partial H^*}{\partial q_i} = H_{q_i} = G_i(\vec{x}^*, \vec{z}^*) = \dot{x}_i.$$

– (3) Recall $q_i = V_{x_i}$. Thus,

$$\frac{dq_i}{dt} = \dot{q}_i = \sum_{j=1}^{s} V_{x_i x_j} \dot{x}_j + V_{x_i t}.$$

– (4) Using $H^*(\cdot) = -V_t$, differentiate with respect to $x_i$:

$$-V_{x_i t} = H^*_{x_i} + \sum_{j=1}^{s} H^*_{q_i} V_{x_i x_j}.$$

$$-V_{x_i t} = H^*_{x_i} + \sum_{j=1}^{s} \dot{x}_j V_{x_i x_j}.$$

$$-H^*_{x_i} = V_{x_i t} + \sum_{j=1}^{s} \dot{x}_j V_{x_i x_j}.$$

Combine (3) and (4),

$$\dot{q}_i = -H^*_{x_i}.$$

## 24.2 Pontryagin Maximum Principal

- Let $\vec{z}^*(t)$ be the choice of instruments that maximize $\int u(\cdot)$ such that (1b) and (1c) hold. Then there exists co-state variables $q_i$ such that:

  – (1) $\vec{z}^*(t)$ maximizes $H(\vec{x}, \vec{z}, \vec{q}, t)$, so:

  $$\frac{\partial H}{\partial z_k} = 0 \; for \; k = 1, \ldots, n,$$

  for an INTERIOR solution.

  – (2) $\dot{q}_i = -\dfrac{\partial H}{\partial x_i}$ evaluted at $\vec{z}^*$.

  – (3) (1b) gives us $\dot{x}_i(t) = G_i(\cdot)$ for $i = 1, \ldots, s$.

- So we have $2 * s$ differential equations so we need $2 * s$ boundry conditions to solve this problem. (1c) gives us $s$ boundry conditions and we will now use the Transversality Conditions (TVC) for the other conditions.

- Assume a finite horizon, $T < \infty$. If there is no scrap value to your states, then $x_i(T) = 0$ for all $i$ so this would be our other $s$ conditions. If there was scrap value then:

$$V(\vec{x}(T), T) = S(\vec{x}(T)) \Rightarrow q_i(T) = \frac{\partial V(\vec{x}(T), T)}{\partial x_i(T)} = \frac{\partial S(\vec{x}(T))}{\partial x_i(T)},\ i = 1, \ldots, s.$$

  And these would be our other $s$ boundry conditions.

- For an infinite horizon, $T = \infty$, define:

$$S(\vec{x}(T)) = \sum_{i=1}^{s} P_i * min\{x_i(T), 0\},$$

  where $P_i >> 0$. So the scrap value of a state variable is multiplied by this very large number which is added to the utility function. If $x_i(T)$ is non-negative, then the scrap value becomes zero. If $x_i(T)$ is negative, we get this huge negative number added on to our objective which is clearly a bad thing. Thus, having this type of scrap value equation induces $x_i(T) \geq 0$. (Not sure why we don't just impose this restriction directly?) So if $x > 0$, $S = 0$, and $\partial S/\partial x = 0$. If $x = 0$, $0 < \partial S/\partial x < P$. Combining these three and using the shadow value for the partial of the scrap value with respect to $x_i$:

$$q_i(T) \geq 0 \quad and \quad q_i(T)x_i(T) = 0\ i = 1, \ldots, s.$$

  Taking the limit,

$$lim_{T\to\infty} q_i(T) \geq 0.$$

$$lim_{T\to\infty} q_i(T)x_i(T) = 0.$$

  And these are the boundry conditions (transversality conditions) in the infinite horizon problem.

# 25 Lecture 25: November 29, 2004

## 25.1 Investment

- When studying the theory of investment, one must distinguish between stock and flow variables. In discrete time, everything is a stock, but with continuous time, we allow for variables "per unit of time." In the Diamond model, $r_t = f'(k_t)$ where $k_t$ was a stock: $k_t = f'^{-1}(r_t)$.

- In the neoclassical theory of investment (firm level), we have production:

$$Y_t = F(K_t, L_t) = K_t^\alpha L_t^{1-\alpha}.$$

  And the firm demands capital until the marginal product of additional capital just equals its cost or:

$$F_K(K_t, L_t) = r_t \Longrightarrow K_t^{Desire} = \frac{\alpha Y_t}{r_t}.$$

  This is again, a stock variable. However, we also introduce a partial adjustment equation:

$$I_t = \mu(K_t^{Desire} - K_t).$$

  Where $I_t$ is the flow of investment per unit of time and $\mu$ reflects the cost of moving towards our desired stock of capital. (Jorgensen 63). In this model, $\mu$ is exogeneous but it is crucial in determining not what the desired level of capital is, but rather how we should get there.

- There are three models which we will consider involving this adjustment to desired capital.

  - (1) Hayashi Model. Installation Function. Define Installed Capacity or installed capital as some function, $\psi(I_t, K_t)$ where,

$$\psi(I_t, K_t) < I_t.$$

    So we invest an amount $I_t$ but only install a fraction of it into usable capital. Assume $\psi_I > 0$, $\psi_{II} < 0$. Define the flow of capital as:

$$\dot{K}_t = \psi(I_t, K_t) - \delta K_t.$$

    And the profit flow:

$$\pi_t = \underbrace{\pi(K_t, L_t)}_{F(\cdot) - w_t L_t} - p_t^k I_t.$$

    Where $\pi(K_t, L_t)$ is operating profits and $p_t^k$ is the exogeneous price of investment (per unit). Assume $w_t$ is also exogeneous and the output price is 1. Here the adjustment costs are all captured in the $\psi$ function.

– (2) Abel and Eberly. Cost of Adjustment Model. Now,

$$\dot{K}_t = I_t - \delta K_t.$$

$$\pi_t = \pi(K_t, L_t) - c(I_t, K_t).$$

Where $c(\cdot)$ is the cost of adjustment function.

– (3) Tobin's $q$. Define:

$$q = \frac{\text{Marginal Market Value of Capital}}{\text{Reproduction Cost}}.$$

So on top, we have the value of the capital (say what the house will be worth), and on the bottom is the cost to build the house. If $q > 1$, then invest. However, we have two problems with this: first, it's difficult to measure marginal values (usually we are measuring average values) and second, even if $q > 1$ and we should invest, how much and how fast?

## 25.2 Hayashi's Model - Installation Function

- A price taking firm used capital and labor to produce output (with price 1) that has value:

$$V(K_t, L_t) = \int_{s=t}^{\infty} e^{-\rho(s-t)}[F(K_s, L_s) - w_s L_s - p_s^k I_s]dt.$$

Thus the firm maximizes the lifetime value of revenues minus costs subject to:

$$\dot{K}_t = \psi(I_t, K_t) - \delta K_t, \quad \psi_I > 0, \psi_{II} < 0, lim_{I\to\infty}\psi_I = 0.$$

- What is the optimal rate of investment? We need the hamiltonian to solve. Denote the present value hamiltonian:

$$H(\cdot) = e^{-\rho t}[F(K_t, L_t) - w_t L_t - p_t^k I_t] + \eta(\psi(I_t, K_t) - \delta K_t).$$

Or the current value hamiltonian: $\tilde{H}(\cdot) = e^{\rho t} H(\cdot)$:

$$\tilde{H}(\cdot) = [F(K_t, L_t) - w_t L_t - p_t^k I_t] + \lambda(\psi(I_t, K_t) - \delta K_t).$$

Where $\lambda = e^{\rho t}\eta$. Our choice variables are $I$ and $L$ and our states are $K$ and $\lambda$.

- FOCs for current value $H$:

$$\tilde{H}_L : F_L(K, L) - w = 0, \quad (11a)$$

$$\tilde{H}_I : -p^k + \lambda\psi_I = 0, \quad (11b)$$

$$\tilde{H}_\lambda :\to \dot{K} = \psi(K, I) - \delta K, \quad (11c)$$

$$\tilde{H}_K :\to \tilde{H}_K = \rho\lambda - \dot{\lambda} \to F_K(K, L) + \lambda(\psi_K(I, K) - \delta) = \rho\lambda - \dot{\lambda} \to$$

$$\rightarrow \dot{\lambda} = \lambda(\rho + \delta - \psi_K(I,K)) - F_K(K,L). \quad (11d)$$

- Note if we used the present value $H$, (note $\eta = e^{-\rho t}\lambda$), the last FOC becomes:

$$\begin{aligned}
H_K :\rightarrow H_K &= -\dot{\eta} \\
e^{-\rho t}F_K(K,L) + \eta(\psi_K(I,K) - \delta) &= -(-\rho e^{-\rho t}\lambda + e^{-\rho t}\dot{\lambda}) \\
e^{-\rho t}F_K(K,L) + e^{-\rho t}\lambda(\psi_K(I,K) - \delta) &= e^{-\rho t}(\rho\lambda - \dot{\lambda}) \\
F_K(K,L) + \lambda(\psi_K(I,K) - \delta) &= \rho\lambda - \dot{\lambda} \\
\dot{\lambda} &= -F_K(K,L) - \lambda(\psi_K(I,K) - \delta) + \rho\lambda \\
\dot{\lambda} &= \lambda(\rho - \delta - \psi_K(I,K)) - F_K(K,L)
\end{aligned}$$

So thankfully, they are the same!!

- Interpretation.

$$(11a) \rightarrow F_L(\cdot) = w \rightarrow L^D = L(w,K).$$

$$(11b) \rightarrow \lambda\psi_I = p^k \rightarrow I^D = I(\lambda/p^k, K).$$

$$(11c) \rightarrow \dot{K} = \psi(I(\lambda/p^k, K), K) - \delta K = \Gamma(K_t, \lambda_t; p^k).$$

$$(11d) \rightarrow \dot{\lambda} = \lambda(\rho + \delta - \psi_K(I(\lambda/p_k, K), K)) - F_K(K, L(w,K)) = \Lambda(K, \lambda; w, p^k).$$

So these last two equations are both first order differential equations that are functions of only states, co-states, and parameters.

- We need two boundry conditions : (1) $K_0$ given. (2) TVC:

$$lim_{t\rightarrow\infty} e^{-\rho t}\lambda_t \geq 0, \quad lim_{t\rightarrow\infty} e^{-\rho t}\lambda_t K_t = 0.$$

- To go further, suppose $\psi_K = 0$, then (11d) becomes:

$$\dot{\lambda} = \lambda(\rho + \delta) - F_K(K,L).$$

Integrating forward ...

$$\dot{\lambda} - \lambda(\rho + \delta) = -F_K(K,L).$$

$$e^{-(\rho+\delta)t}(\dot{\lambda} - \lambda(\rho + \delta)) = -e^{-(\rho+\delta)t}F_K(K,L).$$

$$e^{-(\rho+\delta)t}\dot{\lambda} - e^{-(\rho+\delta)t}\lambda(\rho + \delta) = -e^{-(\rho+\delta)t}F_K(K,L).$$

Note LHS can be rewritten as the derivative:

$$\frac{de^{-(\rho+\delta)t}\lambda(t) + \phi}{dt} = -e^{-(\rho+\delta)t}F_K(K,L).$$

$$(e^{-(\rho+\delta)t}\dot{\lambda(t)} + \phi) = -e^{-(\rho+\delta)t}F_K(K,L).$$

Integrate both sides from $s = t$ to $s = \infty$,

$$\int_{s=t}^{s=\infty} (e^{-(\rho+\delta)t}\dot{\lambda}(t) + \phi)ds = -\int_{s=t}^{s=\infty} e^{-(\rho+\delta)s} F_K(K, L)ds.$$

$$e^{-(\rho+\delta)s}\lambda(s)\Big|_{s=t}^{s=\infty} = -\int_{s=t}^{s=\infty} e^{-(\rho+\delta)s} F_K(K, L)ds.$$

$$-e^{-(\rho+\delta)t}\lambda(t) = -\int_{s=t}^{s=\infty} e^{-(\rho+\delta)s} F_K(K, L)ds.$$

$$\lambda(t) = e^{(\rho+\delta)t}\int_{s=t}^{s=\infty} e^{-(\rho+\delta)s} F_K(K, L)ds.$$

$$\lambda(t) = \int_{s=t}^{s=\infty} e^{-(\rho+\delta)(s-t)} F_K(K, L)ds.$$

Which is the PDV of one more unit of $K_t$.

- To interpret (11b), $\lambda\psi_I = p_k$, consider two special cases:
  - (1) $\lim_{I\to\infty}\psi_I = 0$. This implies there are adjustment costs to more investment. This means we will only have infinite investment when $p^k$ is zero.
  - (2) $\psi(I) = I$. This implies NO adjustment costs. Thus, $\lambda = p^k$. Suppose $K_0$ is low so that $\lambda_0 > p^k$. This means that we should have an infinite rate of investment until $\lambda = p^k$. Thus the presence of adjustment costs are crucial to the model.
- Finally, (11b):

$$I^D = I(\lambda/p^k, K).$$

This first term in the parens is the value of another unit of capital (investment) divided by the reproduction cost. This is exactly Tobin's Q!! Thus,

$$I^D = I(q_t, K_t).$$

# 26 Lecture 26: December 1, 2004

## 26.1 More on the Hayashi Model

- Recall our two first order differential equations from last time:

$$\dot{K} = \psi(I(\lambda/p^k), K) - \delta K, \quad (1)$$

$$\dot{\lambda} = (\rho + \delta + \psi_K(I(\cdot), K)) - F_K(K, L(\cdot)). \quad (2)$$

- See G-26.1 for the phase diagram.
- We assume $w = p^k = 1$ and that $\psi(\cdot)$ is only a function of $I$, not $K$.
- For $\dot{K}$,

$$\dot{K} = 0 \Rightarrow \psi(I(\lambda)) = \delta K.$$

Differentiate with respect to $\lambda$:

$$\psi_I I_\lambda = \delta K_\lambda \Rightarrow K_\lambda = \frac{\psi_I I_\lambda}{\delta}.$$

Note from our FOC, $\lambda \psi_I = p_k = 1$. Differentiate this with respect to $\lambda$,

$$\psi_I + \lambda \psi_{II} I_\lambda = 0.$$

$$I_\lambda = -\frac{\psi_I}{\lambda \psi_{II}} > 0.$$

So,

$$K_\lambda = \frac{\psi_I I_\lambda}{\delta} > 0.$$

So the $\dot{K} = 0$ line is upward sloping. Fixing $\lambda$ and noting that increasing $K$ will decrease $\dot{K}$ means that $K$ is falling to the right of the $\dot{K} = 0$ locus.

- For $\dot{\lambda}$,

$$\dot{\lambda} = 0 \Rightarrow (\rho + \delta)\lambda = F_K(K, L(w, K)).$$

(Assuming $\psi_K = 0$). Differentiate with respect to $\lambda$:

$$\rho + \delta = F_{KK} K_\lambda.$$

(Assuming that $L$ is not a function of $\lambda$). Thus,

$$K_\lambda = \frac{\rho + \delta}{F_{KK}} < 0.$$

So the $\dot{\lambda} = 0$ line is downward sloping. Fixing $\lambda$ and noting that increasing $K$ will increase $\dot{\lambda}$ means that $\lambda$ is rising to the right of the $\dot{\lambda} = 0$ locus.

- So using these loci of stability and the directions of motion, we can draw in the stable arm (and unstable arm) which means this system is saddle-path-stable.
- Given a $K_0$, there is only one $\lambda_0$ which will converge to the steady state. Since the TVC holds, we must be on this stable arm (SA) !! Otherwise, we would diverge to infinity and this would be a bad thing.

**Policy Experiment 1**

- Suppose at time $t$, $\rho$ jumps permanently and unexpectedly to $\rho' > \rho$. Starting point: steady state.
- See G-26.2. If $\rho$ jumps up, holding $K$ constant, $\lambda$ must fall via equation 2. So we move from $E_1$ to $B$ to $E_2$. Note that $K$ cannot immediately jump since there are adjustment costs to investment.

**Policy Experiment 2**

- Suppose at time $t$, $\rho$ jumps permanently and unexpectedly to $\rho' > \rho$. Starting point: Off SS.
- See G-26.3. We move from $E_1$ to $A$ to $E_2$. Same story as in experiment 1.

**Policy Experiment 3**

- Suppose at time $t_1$, $\rho$ jumps unexpectedly though temporarily to $\rho' > \rho$. It is known that it will jump back to $\rho$ at time $t_2$. Starting point: Off SS.
- See G-26.4. We move from $E_0$ to $E_1$ to $A$ to $B$ to $E_2$. In this case, at time $t_2$, $\lambda$ jumps to point ($A$) such that at time $t_2$, we have converged back to the original stable arm (point $B$). We never actually reach SA2. Between time $t_1$ and $t_2$ we are off any stable arm but converging back to SA1. Note the distance of the first jump at $t_1$ is completely determined by when $\rho$ will move back to its original level. It must be exactly such that by $t_2$, we are back on SA1.

**Policy Experiment 4**

- Suppose at time $t_1$, an announcment is made that $\rho$ will jump permanently to $\rho' > \rho$ at time $t_2$. Starting point: Off SS.
- See G-26.5. We move from $E_0$ to $E_1$ to $A$ to $B$ to $E_2$. So at time $t_1$, we also jump part of the way towards SA2, but only enough that between $t_1$ and $t_2$ we are off any SA and we end up on SA2 at time $t_2$. Similar to experiment 3.
- Finally, note the shadow price of capital is the value of capital if the firm invests optimally.

# 27 Lecture 27: December 6, 2004

## 27.1 Linearization of Hayashi

- Recall our two dynamic first order differential equations in the Hayashi Model:

$$\dot{K} = -\delta K + \psi(I(\lambda)).$$

$$\dot{\lambda} = -F_K(K, L) + (\rho + \delta)\lambda.$$

- Previously, we found that the system was saddle path stable (G-27.1) by looking at the phase diagram but we could also solve for the linearized system to determine stability properties.
- Via the Grobman Hartman theorem, we can linearize around a hyperbolic equilibrium such that:

$$\dot{x}(t) = DF[x(t)]\Big|_{x(t)=x^{ss}} * [x(t) - x^{ss}].$$

- Thus, we have:

$$\dot{K} = -\delta(K - K^{ss}) + \psi_I I_\lambda(\lambda - \lambda^{ss}).$$

$$\dot{\lambda} = -F_{KK}(K - K^{ss}) + (\rho + \delta)(\lambda - \lambda^{ss}).$$

  Or in matrix form:

$$\begin{bmatrix} \dot{K} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -\delta & \psi_I I_\lambda \\ -F_{KK} & \rho + \delta \end{bmatrix} \begin{bmatrix} K - K^{ss} \\ \lambda - \lambda^{ss} \end{bmatrix}.$$

  Suppose there are vectors $x$ such that $\dot{\vec{x}} = A\vec{x} = \mu\vec{x}$ where $\mu$ are the eigenvalues. Need to solve for $\mu$. It can be shown that the eigenvalues satisfy:

$$\mu^2 - \mu Tr(A) + Det(A) = 0$$

  Where $A$ is the matrix of coefficients in the linearized system. Solving this equation, we see that the eigenvalues for our system are real, distinct and have opposite sign. Thus we have saddle point stability with no cycles. This is the same result we found from the phase diagram.

## 27.2 Abel and Eberly

- We include 3 types of adjustment costs:
  - (1) Purchase and Sales costs. Firms buy at $P_k^+$ and sell at $P_k^-$. Assume $P_k^+ I$ and $P_k^- I$ are non-decreasing in $I$. So the firm faces different costs of buying and selling capital.
  - (2) Specific adjustment costs to installing capital.
  - (3) Fixed Costs.

- Total cost of investment is therefore:

$$\nu C(I, K),$$

where $\nu = 1$ if the firm invests, $\nu = 0$, if there is no investment.

- Note that $C(0, K) > 0$ if there are fixed costs. If you choose to invest, but then invest nothing (!?), then you still face the fixed costs. However $\nu C(I, K) = 0$ if $\nu = 0$, ie if you know from the beginning there will be NO investment.

- Also $C_I(0, K)^+ \geq C_I(0, K)^-$, so the marginal cost of purchasing new capital is at least as big as the proceeds from selling that capital. Thinking of buying a car and then immediately reselling it.

- Dynamics of capital:

$$\dot{K}_t = I_t - \delta K_t.$$

- Firm choose $\nu$, and $I$ to maximize:

$$V(K_t) = Max_{I_{t+s}, \nu_{t+s}} \int_{s=0}^{\infty} e^{-rs}(\pi(K_{t+s}) - \nu_{t+s} C(I_{t+s}, K_{t+s}))ds.$$

Where $\pi(\cdot)$ is output minus labor costs. Define $q = \frac{\partial V}{\partial K}$.

- We could solve via a Hamiltonian, but note that at every point, $V$ must satisfy:

$$rV(K) = Max_{I,\nu}\ \pi(K) - \nu C(I, K) + \frac{dV}{dt}.$$

And this is an arbitrage condition where we have the return on the value function on the left and the value of holding capital (the value of holding the firm) on the right. Thus, rewriting $dV/dt = dV/dK * dK/dt$,

$$rV(K) = Max_{I,\nu}\ \pi(K) - \nu C(I, K) + q(I - \delta K). \quad (33)$$

- Step 1. Solve (33) for $\nu = 1$. Find optimal investment if we do indeed invest. So we would like to maximize:

$$Max_I\ q(I - \delta K) - C(I, K).$$

FOC:

$$C_I(I^*(q, K), K) = q \ \ for \ \underbrace{q < C_I(0, K)^- \ or \ q > C_I(0, K)^+}_{Range\ of\ Action}. \quad (34a)$$

$$I^*(q) = 0 \ \ for \ \underbrace{C_I(0, K)^- \leq q \leq C_I(0, K)^+}_{Range\ of\ Inaction}. \quad (34b)$$

This means we satisfy the first FOC but only in the range of action noted. Since $q$ is the marginal value of capital, we invest (buy capital) if this value is greater than the purchase price of capital and we disinvest (sell capital) when unbolting and selling our current capital is worth more than holding on to it. If we are inside this range, then we do nothing.

- Step 2. What is the optimal $\nu$? Define $\phi(q, K)$ to be the maximized value of $qI - C(I, K)$. Then,

$$\phi(q, K) = qI^* - \nu C(I^*, K) \; for \; \nu = 1.$$

$$\phi(q, K) = qI^* - \nu C(I^*, K) = 0 \; for \; \nu = 0.$$

- See G-27.2. We know that if $\nu = 0$ and $I^* = 0$, then $\phi(q, K) = -C(0, K) < 0$ because of fixed costs. This is the flat region of the graph. Outside of this region of inaction, $\phi(\cdot) \geq -C(0, K)$ because we can do no worse than $-C(0, K)$ by not investing. Hence the $\phi$ curve must slope up outside of the flat region and it must cross the $\phi = 0$ line on both sides (it can be shown). Finally, since $\phi(q)$ is continuous, there must be a section of the $\phi$ curve that is outside our region of inaction but still negative. It cannot immediately jump to 0 outside the region. Thus in the end:

$$q > q_2 \Longrightarrow I > 0,$$

$$q < q_1 \Longrightarrow I < 0,$$

$$q_1 \leq q \leq q_2 \Longrightarrow I = 0.$$

Which means,

$$\nu = 1 \Longleftrightarrow \phi > 0,$$

$$\nu = 0 \Longleftrightarrow \phi \leq 0.$$

Brilliant.

# 28 Lecture 28: December 8, 2004

## 28.1 Money: Cash-In-Advance (CIA) Model

- We look to model the use of money and not an explanation of why people use it to begin with.
- We consider two models with and without uncertainty.

**Model Under Certainty**

- We invoke the CIA constraint because consumers need money to buy goods. Hence:

$$P_t C_t \leq M_t.$$

Where $M_t$ is the demand for money balances at time $t$. Note this is different than a budget constraint as it only involves your cash purchases.

- The idea is similar to the Lucas "Tree" concept where consumers each own a fraction $s_t$ of a tree which produces fruit, or income, of $x_t$ each period, itself a random variable. $\overline{M}_t$ is the money supply of the economy which grows at rate $\omega$.
- The sequencing of events within a period is very important for the model so we formalize this here:
  - Period (1): Consumer begins period $t$ with assets $M_t$ and $s_t$ carried over from last period.
  - Period (2): The consumer buys goods at price $P_t$ such that $P_t C_t \leq M_t$. With certainty, the consumer knows the value of $x_t$ and $\overline{M}_t$ so the price is implied: $P_t(x_t, \overline{M}_t)$. The CIA constraint will either not bind, in which case the consumer has brought over too much cash from the previous period, or it will bind, which does not mean the consumer brought over exactly what he needed, but rather, he probably does not have quite enough cash to meet his expenditures. This is because $x_t$ is usually uncertain, but here $P_t C_t = M_t$ exactly.
  - Period (3): The consumer receives incomes $x_t$ (known 1 period in advance). Income consists of a) dividends $s_t P_t x_t$ which he receives in cash and b) a lump sum transfer of money, $\overline{M}_{t+1} - \overline{M}_t = (\omega_t - 1)\overline{M}_t$, via a helicopter drop.
  - Period (4): Asset markets operate here and the consumer chooses $M_{t+1}$ and $s_{t+1}$ to be carried over.
- NOTES: The CIA constraint is important because the consumer needs cash in period 2 for purchases but does not receive his current income until period 4. Hence he must keep some money on hand from the previous period to have enough for current purchases. The share, $s_t$, has a certain return, via dividends, but money has no explicit return. Money's value comes through its value of future consumption.

- In equilibrium (not shown), $s_t = 1$, $C_t = x_t$, which implies:

$$P_t = \frac{M_t}{x_t}.$$

This also implies of velocity of money of:

$$V_t = \frac{P_t x_t}{M_t} = \frac{M_t x_t}{x_t M_t} = 1.$$

This doesn't sound reasonable. Thus we move to uncertainty.

**Model Under Uncertainty**

- Now $x_t$ and $\omega_t$ are stochastic with:

$$\overline{M}_{t+1} = \omega_t \overline{M}_t.$$

Let our vector of random variables be:

$$z_t = \{x_t, \omega_t\}.$$

$z_t$ is unknown in periods 1 and 2 and is only realized in period 3 and 4.

- Consumer's Problem:

$$Max\ E_t \sum_{s=t}^{\infty} \beta^{s-t} U(c_s), \quad (1)$$

subject to:

$$\underbrace{M_{t+1} + Q_t s_{t+1}}_{Carried\ Over} = \underbrace{M_t + Q_t s_t}_{Money\ +\ Shares} + \underbrace{P_t x_t s_t + (\omega_t - 1)\overline{M}_t}_{Stage\ 3\ Income} - P_t C_t, \quad (2)$$

$$\underbrace{P_t C_t \leq M_t}_{CIA}. \quad (3)$$

Where $Q_t$ is the dollar price of shares of the tree.

- Next define, $\pi = 1/P$, $q = Q/P$, $\hat{M} = M_{t+1}$, $m = M/P$, $\hat{s} = s_{t+1}$. Rewrite constraints as:

$$c + \pi\hat{M} + q\hat{s} = \underbrace{\pi M + (q + x)s + \pi(\omega - 1)\overline{M}}_{Wealth=W}, \quad (2a)$$

$$c \leq \pi M. \quad (3a)$$

- Our value function becomes:

$$V(W, M, z, \overline{M}) = Max_{c,\hat{M},\hat{s}} \left\{ u(c) + \mu(\pi M - C) + \lambda(W - C - \pi\hat{M} - q\hat{s}) + \beta \int_{\hat{z}} V(\hat{w}, \hat{M}, \hat{z}, \hat{\overline{M}}) dF(\hat{z}) \right\}.$$

Where $dF(\hat{z}) = f(\hat{z})d\hat{z}$.

- FOCs in equilibrium ($C_t = x_t$, $M_t = \overline{M}_t$, $M_{t+1} = \overline{M}_{t+1} = \omega_t \overline{M}_t$, and $s_t = 1$):

$$(a) u'(x) = \lambda(z) + \mu(z).$$

$$(b) \lambda\pi = \beta E[\hat{\lambda}\hat{\pi} + \hat{\mu}\hat{\pi}].$$

$$(c) \lambda q = \beta E[\hat{\lambda}\hat{q} + \hat{\lambda}\hat{x}].$$

$$(d) x \leq m.$$

- Explanation of FOC (d):

$$m > x \Rightarrow \mu = 0 \text{ and } \frac{M}{P} > x \Rightarrow M > Px \Rightarrow M > Pc,$$

so the value of money is 0 since we already have enough to cover our consumption.

$$m = x \Rightarrow \mu > 0 \text{ and } \frac{M}{P} = x \Rightarrow M = Px \Rightarrow M = Pc,$$

so the value of money is positive since we face a binding CIA constraint.

- Explanation of FOC (a): Without a CIA constraint, $u'(x) = \lambda(z)$, or the marginal utility of consumption equals the value or marginal wealth, our usual Euler. With the CIA constraint, $u'(x) = \lambda(z) + \mu(z)$, so there is an additional cost of consuming today – less money on hand for tomorrow's purchases!

- Explanation of FOC (b,c): These FOCs are asset pricing euler equations, one for shares of our tree and one for the utility value of money. Rewrite (c) as:

$$\lambda = \beta \frac{1}{q} E[\hat{\lambda}\hat{q} + \hat{\lambda}\hat{x}].$$

$$\lambda = \beta E[\hat{\lambda}(\frac{\hat{q}}{q} + \frac{\hat{x}}{q})].$$

$$\lambda = \beta E[\hat{\lambda} \underbrace{(1 + \frac{\hat{q} - q}{q} + \frac{\hat{x}}{q})}_{1+r_t}].$$

So we have the shadow value of wealth on the left and the discounted expected return on the right where the return is made up of the capital gain and the dividend payout.

# Final Review

## 28.2 Models

### Solow

- $k_{t+1} = s/(1+n)f(k_t)+(1-\delta)/(1+n)k_t \Rightarrow f(k^{ss})/k^{ss} = (n+\delta)/s \Rightarrow f'(k^{ss}) = (n+\delta)/s.$

### Samuelson - 1958

- Overlapping generations with a perishible piece of chocolate. Similar to a social contracts.

### Diamond OLG - 1965

- Cap Accum: $F(K_t, L_t) - (1-\delta)K_t = C_t + K_{t+1}$. Total consumption: $L_t C_t^y + L_{t-1} C_t^o$.
- Steady state: $f(k) - (n+\delta)k = C^y + C^o/(1+n) \Rightarrow f'(k) = n + \delta, GR.$
- Competitive economy requires:

$$r_t = f'(k_t) - \delta.$$

$$w_t = f(k_t) - k_t f'(k_t).$$

- Dynamics of $k$:

$$(1+n)k_{t+1} = s(w_t, r_{t+1}) = s(f(k_t) - k_t f'(k_t), f'(k_{t+1}) - \delta).$$

- We may end up with more capital than the golden rule - current generation does not take future generation's utility into account (dynamic inefficiency).

### Optimal Growth - Infinite Horizon

- Tech constraint:

$$f(k_t) + (1-\delta)k_t = c_t + (1+n)k_{t+1}.$$

- Modified GR: $f'(k) = (1+n)/\beta - 1 + \delta$. Additional discount factor of future consumption.

### Romer's R and D

- $Y_t = K_t^{\alpha}[(1-a_L)LA_t]^{1-\alpha}$. $\dot{A} = Ba_L L A_t$.
- Technological change is now endogeneous.
- Countries should be able to catch up.

**Arrow's Learning by Doing**

- $Y_t = K_t^{\alpha}[A_t L_t]^{1-\alpha}.A_t = BK_t^{\phi}$.
- Cap Accum: $\dot{K}_t = sY_t - \delta K_t$.
- Countries should be able to catch up.
- Countries with higher population growth would have higher technological growth.
- Ignoring the skill factor of labor and the ability to adopt technologies that are available.

**Arariadis/Drazen Human Capital Model**

- $x_{t+1}^i = x_t^i * h(\tau_t, \overline{x}_t)$.
- Agent's maximize:
$$I = (1-\tau_t^i)w_t x_t^i + \frac{w_{t+1}x_{t+1}^i}{R_{t+1}}.$$
- FOC implies: $R_{t+1} \geq \frac{w_{t+1}}{w_t} h_{\tau}(\tau, x)$.
- Simplification: $h = 1 + \gamma(\overline{x})\tau^i$.
- Separating equilibrium if $x_0 < x^*$, a threshold level of human capital. Training and no-training equilibrium might explain why some countries get stuck.

**Time Inconsistency**

- Rep Agent leads to agent not internalizing the gov't budget constraint. TIC solution. With a single agent, we don't have this since they internalize. In general:
$$U^{Command} \geq U^{TIC} \geq U^{Commit} \geq U^{TC}.$$
- Timing does not necessarily cause TIC (as in the single agent problem) it can (as in the rep agent problem). Small atomistic consumer not internalizing the goverment's budget is really what causes the TIC.

**Richardian Equivalence**

- We we just reallocate the taxes and spending over an agents lifetime, there will be no effect on his utility. However, if some of the benefits come before the costs, then RE breaks down. Tax versus debt financing is completely the same as long as the PDV of government spending is unchanged. This is only true for lump sum taxes, distortionary taxes cause RE to fail.

**Continuous Time Optimization**

- Consumer's problem:

$$Max_{c_t}\int_{t=0}^{T} e^{-\rho t}u(c_t)dt, \quad \dot{k}(t) = f(k) - \delta k_t - c_t, \quad k_0 \; given.$$

- Present value $H$ and FOCs:

$$H = e^{-\rho t}u(c_t) + \eta(f(k) - \delta k_t - c_t).$$

$$H_c = 0, \quad H_\eta \Rightarrow \dot{k}, \quad H_k \Rightarrow H_k = -\dot{\eta}.$$

- Current value $\tilde{H}$ and FOCs:

$$\tilde{H} = u(c_t) + \underbrace{e^{\rho t}\eta}_{\lambda}(f(k) - \delta k_t - c_t).$$

$$\tilde{H}_c = 0, \quad \tilde{H}_\lambda \Rightarrow \dot{k}, \quad \tilde{H}_k \Rightarrow \tilde{H}_k = \rho\lambda - \dot{\lambda}.$$

**Investment - Hayashi**

- Installed Capacity function $\psi(I_t, K_t) < I_t$.
- Flow of capital: $\dot{K}_t = \psi(I_t, K_t) - \delta K_t$.
- Profit flow: $\pi_t = F(\cdot) - w_t L_t - p_t^K I_t$.
- Tobin:

$$q = \frac{\text{Marginal Market Value of Capital}}{\text{Reproduction Cost}}.$$

- Result: $K_\lambda > 0$ along $\dot{K} = 0$, Opposite for $\dot{\lambda}$. Saddle path stable. $K$ is a crawler. $\lambda$ is a jumper.
- Linearization of Hayashi:

$$\dot{K} = -\delta K + \psi(I(\lambda)).$$

$$\dot{\lambda} = -F_K(K, L) + (\rho + \delta)\lambda.$$

$$\dot{K} = -\delta(K - K^{ss}) + \psi_I I_\lambda(\lambda - \lambda^{ss}).$$

$$\dot{\lambda} = -F_{KK}(K - K^{ss}) + (\rho + \delta)(\lambda - \lambda^{ss}).$$

- It can be shown that the eigenvalues satisfy:

$$\mu^2 - \mu Tr(A) + Det(A) = 0$$

**Abel and Eberly**

- We include 3 types of adjustment costs:
  - (1) Purchase and Sales costs. Firms buy at $P_k^+$ and sell at $P_k^-$. Assume $P_k^+ I$ and $P_k^- I$ are non-decreasing in $I$. So the firm faces different costs of buying and selling capital.
  - (2) Specific adjustment costs to installing capital.
  - (3) Fixed Costs.
- Total cost of investment is therefore:

$$\nu C(I, K),$$

where $\nu = 1$ if the firm invests, $\nu = 0$, if there is no investment.

- Dynamics of capital:

$$\dot{K}_t = I_t - \delta K_t.$$

- Firm choose $\nu$, and $I$ to maximize:

$$V(K_t) = Max_{I_{t+s}, \nu_{t+s}} \int_{s=0}^{\infty} e^{-rs}(\pi(K_{t+s}) - \nu_{t+s} C(I_{t+s}, K_{t+s}))ds.$$

- Thus in the end:

$$q > q_2 \Longrightarrow I > 0,$$
$$q < q_1 \Longrightarrow I < 0,$$
$$q_1 \leq q \leq q_2 \Longrightarrow I = 0.$$

Which means,

$$\nu = 1 \Longleftrightarrow \phi > 0,$$
$$\nu = 0 \Longleftrightarrow \phi \leq 0.$$

Brilliant

## 28.3 PS Notes

- Solow Dynamic Equation:

$$(1 + n)k_{t+1} = sf(k_t) + (1 - \delta)k_t.$$

- Solow SS:

$$\frac{f(k^{ss})}{k^{ss}} = \frac{n + \delta}{s}.$$

- Euler: $F(K, L) = F_K * K + F_L * L$.

- Competitive rates:

$$w_t = f(k) - kf'(k).$$

$$r_t = f'(k) - \delta.$$

- Capital Market Equilibrium in Diamond OLG:

$$K_{t+1} = L_t s_t \Longrightarrow (1+n)k_{t+1} = s(\cdot).$$

$$(1+n)k_{t+1} = s(f(k_t) - k_t f'(k_t), f'(k_t) - \delta).$$

- Present Value Hamiltonian:

$$H = e^{-\rho t}u(c_t) + \mu_t(constraint\ in\ K). \Longrightarrow H_K(t) = -\dot{\mu}(t).$$

- Current Value Hamiltonian:

$$\tilde{H} = e^{\rho t}H = u(c_t) + \underbrace{e^{\rho t}\mu_t}_{\lambda_t}(constraint\ in\ K). \Longrightarrow H_K(t) = \rho\lambda(t) - \dot{\lambda}(t).$$