Microeconomics I
Michaelmas Term

Matthew Chesnes
The London School of Economics

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1 Week 1

1.1 The Theory of the Firm

- Objective: Maximize Profits by choosing output and input quantities to generate production plans subject to technology constraints.
- Production Plan: $z = (z_1, ..., z_n)$ of net inputs. If z_i is positive, z_i is an output of production and if z_i is negative, z_i is an input into production.
- Choose z from a production set, Y, where Y is the set of all technologically feasible production plans.
- Case of one output, the production plan might look like $(y, -x)$ where x is a vector of inputs such that $x = (x_1, ..., x_n)$ is a nonnegative vector.
- Define: Input Requirement Set, $V(y)$.

  $$V(y) = \{x \in \mathbb{R}^n_+ : (y, -x) \in Y\}.$$

  In words, all combinations of inputs that allow the firm to produce y units of output, perhaps more.
- [**G-1.1**] Define: An Isoquant, $Q(y)$.

  $$Q(y) = \{x \in \mathbb{R}^n_+ : x \in V(y) \text{ and } x \notin V(y') \text{ when } y' > y\}.$$

  In words, all combinations of inputs that product exactly y units of output, but nothing more.
- Define: The Production Function: The maximum output obtainable from the input combination, $x = (x_1, ..., x_n)$.
- Cobb-Douglas Production Function: $y = x_1^\alpha x_2^\beta$.
- In general, $y = f(x_1, x_2, ..., x_n)$.
- Consider the two input case where $y = f(x_1, x_2)$. Totally differentiating,

  $$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2.$$

- Along an isoquant, $Q(y)$, $dy = 0$, so

  $$\frac{\partial f}{\partial x_1} dx_1 = -\frac{\partial f}{\partial x_2} dx_2.$$

  Thus,

  $$\frac{dx_2}{dx_1} = -\frac{\frac{\partial f}{\partial x_1}}{\frac{\partial f}{\partial x_2}}.$$

  Which is defined as the "Marginal Rate of Technical Substitution" (MRTS).

- Consider the case of linear technology, $y = ax_1 + bx_2$. Thus the technical rate of substitution is $-\frac{a}{b}$. So the two inputs are perfect substitutes.

- Consider the case of fixed proportion technology or a Leontief production function, $y = min\{ax_1, bx_2\}$. The technical rate of substitution is undefined and thus the two inputs are perfect compliments.

- Returns to Scale.

  - $f(tx) = tf(x) \Rightarrow$ Constant Returns to Scale (CRS).
  - $f(tx) > tf(x) \Rightarrow$ Increasing Returns to Scale (IRS).
  - $f(tx) < tf(x) \Rightarrow$ Decreasing Returns to Scale (DRS).
  - For a Cobb-Douglas production function, DRS if $\alpha + \beta < 1$, CRS if $\alpha + \beta = 1$, and IRS if $\alpha + \beta > 1$.

- Homogenous Function: a function $f(x)$ is homogenous of degree k if $f(tx) = t^k f(x) \; \forall \; t > 0$ and $x \in$ the domain. The Cobb-Douglas production function is homogeneous of degree $\alpha + \beta$.

- Homothetic Function: A function $f(x)$ is homothetic if it is a monotonically increasing transformation of a homogenous function of degree 1. That is, there exists an increasing function g, and a homogeneous of degree 1 function $h(x)$, such that $f(x) = g(h(x))$.

- Profit Maximization in a perfectly competitive market.

  - Let p be the price of output and $w = (w_1, ..., w_n)$ be the vector of input prices.
  - The objective of the firm is to choose a production plan $(y, -x)$ to maximize profits,
  $$\pi = py - wx,$$
  subject to a constraint,
  $$(y, -x) \in Y.$$
  [Note: $wx = w_1x_1 + w_2x_2 + ... + w_nx_n$. The constraint simply specifies that $(y, -x)$ is technologically feasible.]

2 Week 2

- The firm's problem is to choose $(y, -x)$ to maximize,

$$\pi = py - wx,$$

subject to,

$$y = f(x).$$

- Substituting the firm is maximizing, $\pi = p(f(x)) - wx$.
- First Order Conditions (FOCs):

$$\frac{\partial \pi}{\partial x_i} \Rightarrow p\frac{\partial f}{\partial x_i} - w_i = 0.$$

$$p\frac{\partial f}{\partial x_i} = w_i.$$

- Thus the value of the marginal product of an input equals the marginal cost of that input.
- Second Order Conditions (SOCs): Let $f_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$. Thus the Hessian matrix of second order derivatives is defined as,

$$H = \begin{bmatrix} pf_{11} & pf_{12} & ... & pf_{1n} \\ pf_{21} & pf_{22} & ... & pf_{2n} \\ ... & ... & ... & ... \\ pf_{n1} & pf_{n2} & ... & pf_{nn} \end{bmatrix}. \tag{1}$$

- Necessary SOC $\Rightarrow$ H matrix must be negative semidefinite.
- Sufficient SOC $\Rightarrow$ H matrix must be negative definite.
- Both of these conditions are LOCAL conditions.
- Solving the FOCs yields $x_i(p, w)$ for $i = 1...n$. These are the unconditional factor demands, or the optimal amount of input i given prices p and x. Thus $f(x(p, w))$ is the amount of output a firm chooses to produce given prices. Thus $f(x(p, w))$ is the firm's supply function. Denote:

$$f(x(p, w)) = y(p, w) \equiv \text{ Firm's Supply Function.}$$

- Profits of the firm:

$$\pi(p, w) = p * y(p, w) - w * x(p, w).$$

Which is the maximum profits a firm can achieve at given prices.

- Both the supply function and the input demand function are homogeneous of degree zero. Thus, for a constant, t,

$$y(tp, tw) = y(p, w).$$

$$x(tp, tw) = x(p, w).$$

- Property of the supply function, y is that it is increasing in the output price, p. The input demand functions are decreasing in the input prices, w.

- Consider a general production function, $f(x_1, x_2)$. The FOCs would be,

$$p\frac{\partial f}{\partial x_1} - w_1 = 0,$$

$$p\frac{\partial f}{\partial x_2} - w_2 = 0.$$

Totally differentiating the FOCs yields,

$$p\frac{\partial^2 f}{\partial x_1^2}dx_1 + p\frac{\partial^2 f}{\partial x_1 \partial x_2}dx_2 - dw_1 = 0,$$

$$p\frac{\partial^2 f}{\partial x_2 \partial x_1}dx_1 + p\frac{\partial^2 f}{\partial x_2^2}dx_2 - dw_2 = 0.$$

Or, in matrix form,

$$\begin{bmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{bmatrix} \begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} dw_1 - \begin{bmatrix} 0 \\ 1 \end{bmatrix} dw_2 = 0. \quad (2)$$

Thus, via cramer's rule,

$$\frac{\partial x_1}{\partial w_1} = \frac{\begin{vmatrix} 1 & pf_{12} \\ 0 & pf_{22} \end{vmatrix}}{\begin{vmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{vmatrix}} = \frac{pf_{22}}{\begin{vmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{vmatrix}}. \quad (3)$$

- Since the first principal minor is negative and the second is positive, the matrix is negative semidefinite and thus all diagonal entries are non-positive so $pf_{22} \leq 0$. Thus,

$$\frac{pf_{22}}{\begin{vmatrix} pf_{11} & pf_{12} \\ pf_{21} & pf_{22} \end{vmatrix}} = \frac{\partial x_1}{\partial w_1} \leq 0. \quad (4)$$

Or the unconditional input demand is downward sloping in prices.

- Warning! Profit maximization only works as described above if there are DRS. If there are CRS or IRS, profits will grow without bounds by buying up more and more inputs.

2.1 Properties of the Profit Function

- $\pi(p, w)$ is non-decreasing in p and is non-increasing in w_i for $i = 1...n$.
  Proof: Let $p' > p$. Thus,

$$\pi(p, w) \leq p'y(p, w) - wx(p, w) \leq \pi(p', w)$$

  This first inequality is just algebra, but the second comes from the fact that π is maximized for (p', w) so using any other (p^*, w^*) will yield lower profits.
  Thus if $p' > p$, Then $\pi(p'w) \geq \pi(p, w)$, or $\pi(p, w)$ is non-decreasing in p.

- $\pi(p, w)$ is homogeneous of degree 1 in prices. Proof: By definition,

$$py(p, w) - wx(p, w) \geq py' - wx' \ \forall \ (y', -x') \in Y,$$

  because the first expression of profits is evaluated at the profit maximizing values of p and w. Multiplying both sides by t,

$$tpy(p, w) - twx(p, w) \geq tpy' - twx' \ \forall \ (y', -x') \in Y$$

  Hence, $(y(p, w), -x(p, w))$ is the production plan that maximizes profits at (tp, tw). Thus $\pi(tp, tw) = t\pi(p, w)$, or π is homogeneous of degree 1.

- $\pi(p, w)$ is convex in prices.
  Proof: Take prices (p, w) and (p', w') and define,

$$(p'', w'') = \lambda(p, w) + (1 - \lambda)(p', w') \text{ for } \lambda \in [0, 1].$$

  Now, by definition of profit maximization,

$$py(p'', w'') - wx(p'', w'') \leq \pi(p, w).$$

  Mulitplying both sides by λ, $(*)$

$$\lambda py(p'', w'') - \lambda wx(p'', w'') \leq \lambda\pi(p, w).$$

  Also by definition of profit maximization,

$$p'y(p'', w'') - w'x(p'', w'') \leq \pi(p', w').$$

  Mulitplying both sides by $(1 - \lambda)$, $(**)$

$$(1 - \lambda)p'y(p'', w'') - (1 - \lambda)w'x(p'', w'') \leq (1 - \lambda)\pi(p', w').$$

  Summing $(*)$ and $(**)$, yields,

$$p''y(p'', w'') - w''x(p'', w'') \leq \lambda\pi(p, w) + (1 - \lambda)\pi(p', w').$$

  Thus,

$$\pi(p'', w'') \leq \lambda\pi(p, w) + (1 - \lambda)\pi(p', w').$$

  So, π is convex. QED.

- $\pi(p, w)$ is continuous on (p, w).
  Proof: QED.

3 Week 3

3.1 Duality

- We have shown thus far that $f(x) \longrightarrow y(p, w), x(p, w) \longrightarrow \pi(p, w)$.
- But can we go the other direction? yes. To do this, we first need the envelope theorem.
- Envelope Theorem: Supose that we maximize a function $g(s, a)$ with respect to s where a is a parameter. We would obtain a solution $s(a)$. Now define the maximized value of the function as $M(a) = g(s(a), a)$. By the chain rule,

  $$\frac{dM(a)}{da} = \frac{\partial g}{\partial s}\frac{\partial s}{\partial a} + \frac{\partial g}{\partial a}.$$

  But since we already defined $s(a)$ to be the maximium solution of g, $\frac{\partial g}{\partial s} = 0$. Thus

  $$\frac{dM(a)}{da} = \frac{\partial g}{\partial a}.$$

- Now consider the profit function,

  $$\pi(p, w) = py(p, w) - wx(p, w).$$

  Thus, by the envelope theorem,

  $$\frac{\partial \pi}{\partial p} = y(p, w).$$

  $$\frac{\partial \pi}{\partial w_i} = -x_i(p, w).$$

  Thus,

  $$\frac{\partial^2 \pi}{\partial p^2} = \frac{\partial y(p, w)}{\partial p} \equiv \text{Slope of the Supply Function.}$$

  The slope of the supply function is positive because π is convex. Also,

  $$\frac{\partial^2 \pi}{\partial w_i^2} = -\frac{\partial x_i(p, w_i)}{\partial w_i} \geq 0.$$

  Thus, the input demands are downward sloping. Finally,

  $$\frac{\partial^2 \pi}{\partial w_i \partial w_j} = -\frac{\partial x_i(p, w)}{\partial w_i} = -\frac{\partial x_i(p, w)}{\partial w_j} = \frac{\partial^2 \pi}{\partial w_j \partial w_i}.$$

  Because the order that you take the partials does not matter (Young's Theorem).

- This process of taking the partials of the profit function with respect to price, p, to find the supply function, and with respect to the input price, w, to find the input demand is called the **Hotelling Lemma**.

3.2 Cost Minimization

- The objective of the firm can also be to choose an input combination $x = (x_1, ..., x_n)$ to minimize wx subject to $y = f(x)$. Thus,

$$L = wx - \lambda(f(x) - y).$$

FOCs:

$$\frac{\partial L}{\partial \lambda} = y - f(x) = 0.$$

$$\frac{\partial L}{\partial w_i} = w_i - \lambda \frac{\partial f}{\partial x_i} = 0 \text{ for } i = 1...n.$$

Thus,

$$w_i = \lambda \frac{\partial f}{\partial x_i} \ \forall \ i.$$

Thus,

$$\frac{w_i}{w_j} = \frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_j}}.$$

Which implies that the slope of the isocost = the slope of the isoquant at equilibrium. [**G-3.1**]

- To determine the SOCs, we must look at the Boarded Hessian. Consider the FOCs:

$$y - f(x) = 0.$$

$$w_i - \lambda \frac{\partial f}{\partial x_i} = 0.$$

Thus take the derivative of all the FOCs with respect to λ and all the inputs, x. Thus,

$$\tilde{H} = \begin{bmatrix} 0 & -f_1 & ... & -f_n \\ -f_1 & -\lambda f_{11} & ... & \lambda f_{1n} \\ -f_2 & -\lambda f_{21} & ... & \lambda f_{2n} \\ ... & ... & ... & ... \\ -f_n & -\lambda f_{n1} & ... & \lambda f_{nn} \end{bmatrix}. \quad (5)$$

- If we solve the FOCs, we find a function $x_i(w, y) \ \forall \ i$. These are the conditional factor demands because they depend on the quantity of output, y. If we sum up all the conditional factor demands multiplied by their costs,

$$wx(w, y) = \sum_{i=1}^{n} w_i x_i(w, y) = C(w, y),$$

Where $C(w, y)$ is the cost function.

- Properties of the Cost Function, $C(w, y)$.
  - $C(w, y)$ is nondecreasing in w_i for $i = 1...n$.
  - $C(w, y)$ is homogeneous of degree 1 in w.
  - $C(w, y)$ is concave in w.
  - $C(w, y)$ is continuous in w.
- Duality of the Cost Function:

$$C(w, y) = \sum_{i=1}^{n} w_i x_i(w, y).$$

Thus, by the envelope theorem,

$$\frac{\partial C(w, y)}{\partial w_i} = x_i(w, y) \; \forall \; i.$$

So by the Shephard's Lemma, taking the partial of the cost function with respect to the input prices yields the respective input demand function. Also,

$$\frac{\partial^2 C(w, y)}{\partial w_i^2} = \frac{\partial x_i(w, y)}{\partial w_i} \equiv \text{Slope of the Conditional Factor Demand.}$$

Since the Cost function is concave, C is negative semidefinite. Thus all diagonal entries are negative. Thus the conditional input demand functions are downward sloping.

- Finally, having obtained now the cost function, we can write it simply as $C(y)$. Thus,

$$\pi = py - C(y).$$

To maximize profits, a firm chooses y to maximize $py - C(y)$, thus the FOC:

$$\frac{\partial \pi}{\partial y} \Rightarrow p - \frac{\partial C}{\partial y} = 0.$$

Or the familiar result,

$$\text{Price} = \text{Marginal Cost.}$$

4 Week 4

- Costs: $C(y) = VC(y) + F$, where VC are the variable costs that depend on the quantity of output and F are the fixed costs. Fixed costs can be divided into Long and Short run fixed costs.
- Average Costs:
$$AC(y) = \frac{VC(y)}{y} + \frac{F}{y} = AVC(y) + AFC(y).$$
- If $AC(y)$ is decreasing in y, then we have economies of scale.
- If $AC(y)$ is increasing in y, then we have diseconomies of scale.
- IRS $\Rightarrow$ Economies of Scale, but NOT necessarily vice versa. Symmetrically, DRS $\Rightarrow$ Diseconomies of Scale but NOT vice versa.
- Marginal Cost: $\dfrac{dC(y)}{dy} = C'(y) \equiv \text{MC}$.
- Since $AC(y) = \dfrac{C(y)}{y}$,
$$AC'(y) = \frac{d}{dy}\left[\frac{C(y)}{y}\right] = \frac{yC'(y) - C(y)}{y^2} = \frac{C'(y)}{y} - \frac{C(y)}{y^2}.$$
Therefore, when $AC'(y) = 0$,
$$\frac{C'(y)}{y} = \frac{C(y)}{y^2}.$$
$$C'(y) = MC(y) = \frac{C(y)}{y}.$$
Or the MC intersects the AC curve at the minimum of the AC. The same can be shown for the AVC : $MC = AVC$ at the minimum of the AVC.
- Short Run Firm Supply.
  - Assume fixed costs are sunk. Thus,
  $$\pi_{SR} = y(P - AVC(y)) - F.$$
  Firms only supply product if $P > AVC(y)$, so graphically **[G-4.1]** the short run supply curve is the MC curve above the minimum of the AVC curve and 0 elsewhere.
  - Short run aggregate supply assuming m firms:
  $$Y(p) = \sum_{i=1}^{m} y_i(p).$$

- Long Run Firm Supply.
  - A firm's supply is positive if $p > min(AC)$.
  - Assume no barriers to entry.
  - Thus if $p > min(AC)$, $\pi > 0$ and firms have an incentive to enter and drive the price down.
  - Also if $p < min(AC)$, $\pi < 0$ and firms have an incentive to exit and bring up the price.
  - Thus, in equilibrium, $p = min(AC)$, or the long run supply curve is perfectly elastic.
- More on Duality. We have shown that via shepherd's lemma, we can go from the cost function to the conditional factor demands by differentiating $C(y)$ with respect to w. But can we go from the conditional factor demands to the technology? Yes, in most cases. By looking at several input and price combinations and determining the equilibrium we begin to trace out an isoquant by looking at the slope of the isocost lines at each point. **This only works if the technology is convex! [G-4.2]**

4.1 Consumer Theory

- Consumer chooses best (based on preferences) available (based on budget constraint) choice.
- Domain of choice: the consumption set, $X \subseteq \mathbb{R}^n_+$. A typical element of X is $x = (x_1, ..., x_n)$, which is a consumption bundle.
- Preferences:
  - $x \succeq^p y \equiv$ "x is weakly prefered to y."
  - $x \succ^p y \equiv$ "x is strictly prefered to y if $x \succeq^p y$ but NOT $y \succeq^p x$."
  - $x \sim^p y \equiv$ "x is indifferent to y if $x \succeq^p y$ and $y \succeq^p x$."
  - Properties of Preferences.
    * Completeness: For any $x, y \in X$, either $x \succeq^p y$ or $y \succeq^p x$, or both. Thus all bundles can be ranked.
    * Transitivity: For $x, y, z \in X$, if $x \succeq^p y$ and $y \succeq^p z$, then $x \succeq^p z$. Also holds for strict preference and indifference.
- Theorem: Suppose the consumption set, X, is finite. If preferences are complete and transitive, then there exists a function $u : X \mapsto \mathbb{R} \ni \ \forall\ x, y \in X, u(x) \geq u(y)$ if and only if $x \succeq^p y$.
  Proof: Suppose, for simplicity, all bundles are ordered by strict preference. Then there exists a most preferred bundle. To show this, proceed by contradiction. Assume there is not a most preferred bundle. Then for any bundle, say x', I can find another bundle,

x'', such that $x'' \succ^p x'$. Starting with x'', there exists a x''' such that $x''' \succ^p x''$. And so on... If there is only a finite number of bundles some bundles must be repeated. But this contradicts transitivity. So there must exist a most preferred bundle, call it x^*. Set $u(x^*) = c_1$, where c_1 is a constant. Now find the second most prefered bundle, y^* and set $u(y^*) = c_2$ where $c_2 < c_1$. This can continue until $u(\cdot)$ is completely defined as described above. QED.

4.2 More Accurate Definiteness Properties

- Test for definiteness as it applies to symmetric matrices (such as the Hessian):
- The matrix is positive definite iff all the leading principal minors are strictly positive.
- The matrix is negative definite iff all the odd-order leading principal minors are strictly negative and all the even-order leading principal minors are strictly positive.
- The matrix is positive semidefinite iff every principal minor (leading and nonleading) is nonnegative.
- The matrix is negative semidefinite iff every principal minor (leading and nonleading) of odd-order is not positive, and every principal minor of even-order is not negative.
- Of course, if we find a matrix to be postive definite then we know it is positive semidefinite and if we find matrix to be negative definite then it is negative semidefinite and we therefore do not need to look at the nonleading principal minors.
- As an example, consider a $3x3$ symmetric matrix where H_1, H_2, and H_3 are the leading principal minors:

$$\text{If } H_1 > 0, H_2 > 0, H_3 > 0,$$
$$\text{then the matrix is Postive Definite.}$$

$$\text{If } H_1 < 0, H_2 > 0, H_3 < 0,$$
$$\text{then the matrix is Negative Definite.}$$

$$\text{If } H_1 > 0, H_2 > 0, H_3 = 0,$$
$$\text{then the matrix is not definite.}$$

In this case the matrix is either Postive Semidefinite or Indefinite. To determine if it is positive semidefinite, we must test the rest of the principal minors (i.e. the non-leading ones). If every principal minor is nonnegative, then the matrix is indeed positive semidefinite.

$$\text{If } H_1 < 0, H_2 > 0, H_3 = 0,$$

then the matrix is either Negative Semidefinite or Indefinite.

Again, we must test all the principal minors to determine if it is negative semidefinite.

$$\text{If } H_1 = 0, H_2 > 0, H_3 = 0,$$

then the matrix is either Postitive Semidefinite,

Negative Semidefinite, or Indefinite.

Again, we must test all principal minors to determine which it is.

- **NOTE:** When considering a Boardered Hessian, everything is the exact opposite. You get negative semi-definitness when the odd principal leading minor determinants are all non-negative and the even are non-positive. This excludes the first, which is alway zero, and the second which is always negative, so you only have to worry about the 3rd and greater. For a negative definite hessian matrix therefore, all principal minor determinants must be strictly positive!

5 Week 5

- Utility is an ordinal concept. Only the relative ranks matter. If $u(x)$ represents the preferences of the consumer, so does $u(x) + c$ because all utility levels just shift by c but the relative position remains the same.

- Any monotonic transformation that preserves the intial utility ranking leaves the utility function unchanged. Thus $[u(x)]^k$ and $ln(u(x))$ are ok transformations.

- Definition: Utility function.

$$u : x \longmapsto \mathbb{R} \ni x \succeq^p y \text{ iff } u(x) \geq u(y) \text{ for any } x, y \in X.$$

- Utility functions are transitive and complete.

- Definition: Indifference Curves: The set of consumption bundles, x, for which $u(x)$ is constant.

- Consider the case of two commodities so that $x = (x_1, x_2)$. An indifference curve is then $u(x_1, x_2) = c$, where c is constant. Totally differentiating,

$$\frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 = 0.$$

  Thus,

$$\frac{dx_2}{dx_1} = \frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}}.$$

  Or,

$$\frac{dx_2}{dx_1} = \text{The slope of the indifference curve.}$$

  Thus the slope of the indifference curve equals the Marginal Rate of Substitution (MRS).

- Example: Cobb-Douglas Utility function. $u(x_1, x_2) = x_1^\alpha x_2^\beta$. Thus,

$$MRS = \frac{\alpha x_1^{\alpha-1} x_2^\beta}{\beta x_1^\alpha x_2^{\beta-1}} = \frac{\alpha x_2}{\beta x_1}.$$

- Example: Perfectly Substitutable Commodities: $u(x_1, x_2) = ax_1 + bx_2$. Thus,

$$MRS = \frac{a}{b}.$$

- Example: Perfectly Complementary Commodities: $u(x_1, x_2) = min\{ax_1, bx_2\}$. Thus,

$$MRS = \text{Undefined.}$$

  This is because the indifference curves are rectangular so we get a corner solution where the slope is undefined.

5.1 The Consumer's Problem

- A consumer chooses a consumption bundle $x = (x_1, x_2, ..., x_n)$ to maximize $u(x_1, x_2, ..., x_n)$ subject to $p_1x_1 + p_2x_2 + ... + p_nx_n \leq m$, where m is the consumer's income.

- Notice that the constraint is written as an inequality. It can be written as a strict equality if we have the following assumption.

- Local NonSatiation (LNS)

  $$\text{For any } x \in X \text{ and } \epsilon > 0, \exists\, y \in X \ni d(x,y) < \epsilon \text{ and } u(y) > u(x).$$

  Where $d(x,y)$ represents the distance between x and y. So in words, it just means that for any bundle of goods, we can find some other bundle close to it where the utility at the new bundle is strictly greater. A little more (possibly very little) can always be better.

- If utility functions are monotonic, you definitely get LNS, but not vice versa. Having LNS does not necessarily mean that $u(x)$ is monotonic. Monotonicity is stronger than LNS.

- If LNS holds, you cannot be in an equilibrium state where you are not spending all available income. You can spend a little bit more and make yourself strictly better off. Thus, repeating this process, you eventually reach your income so the consumer's problem changes.

- Consumer's Problem assuming LNS. Choose $x \in X$ to maximize,

  $$u(x_1, x_2, ..., x_n),$$

  subject to,

  $$p_1x_1 + p_2x_2 + ... + p_nx_n = m.$$

  Or in vector notation, maximize:

  $$u(x) \text{ subject to } px = m,$$

  where p is a price vector.

- Solving the Consumer's Problem, via the LaGrangian,

  $$\mathbb{L} = u(x) + \lambda(m - px).$$

- FOCs:

  $$\frac{\partial L}{\partial \lambda} = m - px = 0.$$

  $$\frac{\partial L}{\partial x_i} = \frac{\partial u}{\partial x_i} - \lambda p_i = 0 \;\forall\, i.$$

- SOCs. Let the first FOC be L_1 and the second set of $FOCs$ be $L_2, L_3, \cdots$.

$$\tilde{H} = \begin{bmatrix} \frac{\partial L_1}{\partial \lambda} & \frac{\partial L_1}{\partial x_1} & ... & \frac{\partial L_1}{\partial x_n} \\ \frac{\partial L_2}{\partial \lambda} & \frac{\partial L_2}{\partial x_1} & ... & \frac{\partial L_2}{\partial x_n} \\ \frac{\partial L_3}{\partial \lambda} & \frac{\partial L_3}{\partial x_1} & ... & \frac{\partial L_3}{\partial x_n} \\ ... & ... & ... & ... \\ \frac{\partial L_n}{\partial \lambda} & \frac{\partial L_n}{\partial x_1} & ... & \frac{\partial L_n}{\partial x_n} \end{bmatrix} = \begin{bmatrix} 0 & -p_1 & ... & -p_n \\ -p_1 & u_{11} & ... & u_{1n} \\ -p_2 & u_{21} & ... & u_{2n} \\ ... & ... & ... & ... \\ -p_n & u_{n1} & ... & u_{nn} \end{bmatrix}. \tag{6}$$

- This boardered Hessian matrix, $\tilde{H}$, should be negative semidefinite because we are looking for a maximum. Thus the leading principal minor determinants starting with the 3^{rd} should alternate in sign and the 3^{rd} should be positive. Note again that this applies only to boardered hessians because we are adding on an extra row and column so everything is backward compared a regular matrix. Note that you start with the 3^{rd} because the 1^{st} is always 0 and the 2^{nd} is always negative.

- Now consider the second set of FOCs:

$$\frac{\partial L}{\partial x_i} = \frac{\partial u}{\partial x_i} - \lambda p_i = 0 \; \forall \; i.$$

  Thus,

$$\frac{\partial u}{\partial x_i} = \lambda p_i.$$

  Therefore comparing commodity i and j, we have

$$\frac{\frac{\partial u}{\partial x_i}}{\frac{\partial u}{\partial x_j}} = \frac{p_i}{p_j}.$$

  Which in words says that the Marginal Rate of Substitution (MRS) is equal to the price ratio, which is the slope of the Budget Constraint (BC).

- In order to have one unique solution to the consumer's problem, there must exist only one tangency between the consumer's indifference curves and budget constraint. Thus, Convex indifference curves or therefore **Quasi-concave** utility functions will avoid any problems with multiple solutions. [**G-5.1**]

- Example using Cobb-Douglas Utility Function: $u(x_1, x_2) = x_1^{\alpha} x_2^{\beta}$. Thus,

$$MRS = \frac{\alpha x_2}{\beta x_1} = \frac{p_1}{p_2}.$$

$$p_2 x_2 = \frac{\beta}{\alpha} p_1 x_1.$$

And from the budget constraint, substituting in,

$$p_1x_1 + p_2x_2 = m.$$

$$p_1x_1 + \frac{\beta}{\alpha}p_1x_1 = m.$$

$$p_1x_1(1 + \frac{\beta}{\alpha}) = m.$$

Thus,

$$x_1 = \frac{m}{p_1}\frac{\alpha}{\alpha + \beta}.$$

And symmetrically,

$$x_2 = \frac{m}{p_2}\frac{\beta}{\alpha + \beta}.$$

These last two equations are called the "Marshallian Demands."

- Examples in the notes show that if one utility function is a monotonic transformation of another, then the Marshallian demands that come out of the maximization problem will be the same. Also diminishing marginal utility is not meaningful in this framework (for some reason). With perfect substitutes, the marshallian demands depend on the prices of the two goods. If one good is cheaper than another, the consumer will just consume all of the cheaper good (and vice versa). If the prices of the goods are equal, then any choice on the budget line is optimal.

6 Week 6

- Perfect Complements Utility Functions: $u(x_1, x_2) = min\{ax_1, bx_2\}$. Let $a = b = 1$ and we know that because of the rectangular utility functions **[G-6.1]**, the optimal solution will be such that $x_1 = x_2$. Thus the budget constraint becomes,

$$p_1 x_1 + p_2 x_2 = p_1 x_1 + p_2 x_1 = m.$$

Thus, the marhshallian demands,

$$x_1 = \frac{m}{p_1 + p_2}.$$
$$x_2 = \frac{m}{p_1 + p_2}.$$

- Quasi Linear Utility Functions: $u(x_1, x_2) = 2\sqrt{x_1} + x_2$, where the first term is clearly nonlinear but the rest is linear. To solve this for the Marshallian demands, do the usual and set MRS = Price Ratio. Thus,

$$\frac{\frac{u}{x_1}}{\frac{u}{x_2}} = \frac{p_1}{p_2} \Rightarrow \frac{x_1^{-1/2}}{1} = \frac{p_1}{p_2}.$$

Thus,

$$x_1 = (\frac{p_2}{p_1})^2.$$

And substituting into the budget constraint,

$$p_1(\frac{p_2}{p_1})^2 + p_2 x_2 = m \Rightarrow x_2 = \frac{m}{p_2} - \frac{p_2}{p_1}.$$

The only problem with this is x_2 might be negative under the right choices for m, p_1, and p_2. Thus Marshallian demands are,

$$x_1 = (\frac{p_2}{p_1})^2 \text{ and } x_2 = \frac{m}{p_2} - \frac{p_2}{p_1} \quad \text{if} \quad \frac{m}{p_2} - \frac{p_2}{p_1} \geq 0.$$
$$x_1 = \frac{m}{p_1} \text{ and } x_2 = 0 \quad \text{if} \quad \frac{m}{p_2} - \frac{p_2}{p_1} < 0.$$

Thus with quasi-linear utility functions, one must be careful because the tangency condition might yield negative demands and thus the solution needs to be adjusted to reflect this. **[G-6.2]**

6.1 Price and "and" Income Changes

- Price Offer Curve: Take your typical utility maximizing setup with the utility curve just tangent to the budget constraint. Then change the price of one of the commodities. Thus the budget line shifts and you will get a new equilibrium at a new utility level. Continuing this process and connecting the equilibrium points yields the Price Offer Curve. If the curve slopes downward in x_i, p_i space then the good is called Ordinary: The lower the price of good i, the more of it that is demanded.

- If the price offer curve slopes upward in x_i, p_i space then the good is called a Giffen Good: The lower the price of good i, the less of it that is demanded. **[G-6.3]**

- Now consider the same utility/budget constraint set up and vary the income level. Record the different equilibriums as you adjust income. The curve that connects all the equilibriums is called the Income Expansion Path. Drawn in x_i, m space it is refered to as an Engel Curve. If the Engel curve slopes upwards in x_i, m space, then the good is called a Normal Good: The higher a consumer's income level, the more of good i that is demanded.

- If the Engel curve slopes downwards in x_i, m space, then the good is called an Inferior Good: The higher a consumer's income level, the less of good i that is demanded. Examples of inferior goods are cheap food products and even cigarettes. **[G-6.4]**

6.2 Indirect Utility Functions

- From Utility maximization, we obtain a vector of marshallian demands $x(p, m)$. Plugging the vector of marshallian demands into the utility function, we obtain the Indirect Utility Function, $v(p, m)$. That is,

$$v(p, m) = u(x(p, m)).$$

- Example of a Cobb Douglas Utility Function: $u(x_1, x_2) = x_1 x_2$. MRS = Price ratio implies,

$$\frac{\frac{\partial u}{\partial x_1}}{\frac{\partial u}{\partial x_2}} = \frac{p_1}{p_2} \Rightarrow \frac{x_2}{x_1} = \frac{p_1}{p_2}.$$

Thus,

$$x_1 = x_2(\frac{p_2}{p_1}).$$

$$x_2 = x_1(\frac{p_1}{p_2}).$$

And substituting into the budget constraint,

$$m = p_1 x_1 + p_2[x_1(\frac{p_1}{p_2})] = 2p_1 x_1.$$

So Marshallian Demands,

$$x_1 = \frac{m}{2p_1}.$$

$$x_2 = \frac{m}{2p_2}.$$

Substituting the Marshallian demands back into the utility functions,

$$v(p, m) = u(x_1(p_1, m), x_2(p_2, m)) = \frac{m}{2p_1}\frac{m}{2p_2} = \frac{m^2}{4p_1 p_2}.$$

6.3 Properties of the Indirect Utility Function

- Non-increasing in p and non-decreasing in m. See notes for simple proof.
- Homogeneous of degree 0 in (p, m). If all prices and income level go up by the same amount, the budget line doesn't shift at all, thus the utility level is constant. $tpx = tm \Rightarrow px = m$.
- Quasi-Convex in Prices.
  Proof: Take two prices vectors, p and p' and consider another price vector,

  $$p'' = \lambda p + (1 - \lambda)p' \text{ with } \lambda \in [0, 1].$$

  Now consider a consumption bundle, $x^* \ni p''x^* \leq m$. Thus x^* is affordable at prices, p''. This implies that x^* must be affordable at either p, p', or both. Thus $px^* \leq m$ or $p'x^* \leq m$ or both. Hence,

  $$v(p'', m) \leq max\{v(p, m), v(p', m)\}.$$

  Which is the definition of quasi-convex in p.
- $v(p, m)$ is continuous if $u(x)$ is continuous. No proof.

6.4 Expenditure Function

- A consumer chooses the bundle x to minimize px subject to $u(x) \geq u$. As a simply analogue to the cost function of the firm, out of this optimisation we obtain an expenditure function with the same properties as the cost function.
- Properties of the Expenditure function $e(p, u)$: 1) Non-decreasing in p; 2) Homogeneous of degree 1 in p; 3) Concave in p; 4) Continuous.
- Just as we took the partial of the cost function with respect to input prices to get the conditional factor demands (via Shephard's lemma), we can take the partial of the expenditure function with respect to prices to get the compensated or Hicksian Demands.

  $$\frac{\partial e(p, u)}{\partial p_i} = h_i(p, u).$$
- Next we have a series of identities which we will use in the coming sections,

  $$\text{Identity 1: } e(p, v(p, m)) = m.$$

  $$\text{Identity 2: } v(p, e(p, u)) = u.$$

  $$\text{Identity 3: } h_i(p, v(p, m)) = x_i(p, m).$$

  $$\text{Identity 4: } x_i(p, e(p, u)) = h_i(p, u).$$

- Taking the total differential of the 2nd identity,

$$\frac{\partial v(p,e(p,u))}{\partial p_i} + \frac{\partial v(p,e(p,u))}{\partial m}\frac{\partial e(p,u)}{\partial p_i} = 0.$$

$$\frac{\partial v(p,e(p,u))}{\partial p_i} + \frac{\partial v(p,e(p,u))}{\partial m} h_i(p,u) = 0.$$

  Evalutated at $u = v(p,m)$,

$$\frac{\partial v(p,m)}{\partial p_i} + \frac{\partial v(p,m)}{\partial m} x_i(p,m) = 0.$$

  Hence solving for the Marshallian demand,

$$x_i(p,m) = -\frac{\frac{\partial v(p,m)}{\partial p_i}}{\frac{\partial v(p,m)}{\partial m}}.$$

  This last Equality is called Roy's Identity named after Roy. You can obtain Marshallian demands from the indirect utility function.

- Example of finding an expenditure functions: Consider a Cobb Douglas Utility function: $u = x_1 x_2$. Thus,

$$v(p,m) = \frac{1}{4}\frac{m^2}{p_1 p_2}.$$

  Solving for m,

$$m = 2\sqrt{p_1 p_2 v(p,m)} = 2\sqrt{p_1 p_2 u} = e(p,u).$$

- We would now like to develop restrictions on the Marshallian demands. Consider,

$$\text{Identity 4: } x_i(p,e(p,u)) = h_i(p,u).$$

- Taking to total differential,

$$\frac{\partial x_i(p,e(p,u))}{\partial p_j} + \frac{\partial x_i(p,e(p,u))}{\partial m}\frac{\partial e(p,u)}{\partial p_j} = \frac{\partial h_i(p,u)}{\partial p_j}.$$

$$\frac{\partial x_i(p,e(p,u))}{\partial p_j} + \frac{\partial x_i(p,e(p,u))}{\partial m} h_j(p,u) = \frac{\partial h_i(p,u)}{\partial p_j}.$$

  Evluated at $u = v(p,m)$,

$$\frac{\partial x_i(p,m)}{\partial p_j} + \frac{\partial x_i(p,m)}{\partial m} x_j(p,m) = \frac{\partial h_i(p,u)}{\partial p_j}.$$

  And this last equality is called the Slutsky Equation. Note that the left side only depends on p and m which are usually observable quantities.

- Simplfying the notation of the Slutsky equation,

$$\Delta_{ij}(p,m) = \frac{\partial h_i(p,u)}{\partial p_j}.$$

Thus in matrix form,

$$\begin{bmatrix} \Delta_{11} & \dots & \Delta_{1n} \\ \vdots & \ddots & \vdots \\ \Delta_{n1} & \dots & \Delta_{nn} \end{bmatrix} = \begin{bmatrix} \frac{h_1}{p_1} & \dots & \frac{h_1}{p_n} \\ \vdots & \ddots & \vdots \\ \frac{h_n}{p_1} & \dots & \frac{h_n}{p_n} \end{bmatrix}. \tag{7}$$

- Notice that the right hand matrix of partials of the hicksian demand equation is equivalent to the matrix of second partials of the expenditure function. Thus since $e(p,u)$ is concave, this matrix must be negative semidefinite. Thus the left hand matrix must also be negative semidefinite. The $\Delta_{ij}(p,m)$ matrix is called the Substitution Matrix.

- Finally, rearranging the terms of the Slutsky equation,

$$\frac{\partial x_1}{\partial p_1} = \underbrace{\frac{\partial h_1}{\partial p_1}}_{Substitution-Effect} - \underbrace{\frac{\partial x_1}{\partial m}}_{Income-Effect} x_1.$$

Since $\frac{\partial h_1}{\partial p_1}$ (the substitution effect) is always negative and $\frac{\partial x_1}{\partial m}$ is positive when x_1 is a normal good, $\frac{\partial x_1}{\partial p_1} < 0$ or demand for good i is downward sloping as would be expected.

- If $\frac{\partial x_1}{\partial m} < 0$ then we have an inferior good. If this income effect term is large enough (negative enough), then $\frac{\partial x_1}{\partial p_1}$ might actually become positive which would imply that the commodity is a giffen good. Thus ALL giffen goods are inferior but not vice versa.

7 Week 7

7.1 Comparative Statics

- The Consumer's Problem: Choose $x \in X$ to maximize $u(x)$ subject to $px = m$. Set up the lagrangian: $L = u(x) + \lambda(m - px)$ which yields FOCs (for the case of 2 commodities),

$$\frac{\partial L}{\partial \lambda} = m - p_1 x_1 - p_2 x_2 = 0.$$

$$\frac{\partial L}{\partial x_1} = \frac{\partial u}{\partial x_1} - \lambda p_1 = 0.$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial u}{\partial x_2} - \lambda p_2 = 0.$$

- Totally differentiating the first FOC for utility maximization yields,

$$dm - x_1 dp_1 - p_1 dx_1 - x_2 dp_2 - p_2 dx_2 = 0.$$

  Or,

$$-p_1 dx_1 - p_2 dx_2 = x_1 dp_1 + x_2 dp_2 - dm.$$

- The second FOC:

$$u_{11} dx_1 + u_{12} dx_2 - \lambda dp_1 - p_1 d\lambda = 0.$$

  Or,

$$-p_1 d\lambda + u_{11} dx_1 + u_{12} dx_2 = \lambda dp_1.$$

- Similarly for the third FOC:

$$u_{22} dx_2 + u_{21} dx_1 - \lambda dp_2 - p_2 d\lambda = 0.$$

  Or,

$$-p_2 d\lambda + u_{21} dx_1 + u_{22} dx_2 = \lambda dp_2.$$

- Written in matrix form then,

$$\begin{bmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u_{11} & u_{12} \\ -p_2 & u_{12} & u_{22} \end{bmatrix} \begin{bmatrix} d\lambda \\ dx_1 \\ dx_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \lambda \\ 0 \end{bmatrix} dp_1 + \begin{bmatrix} x_2 \\ 0 \\ \lambda \end{bmatrix} dp_2 + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} dm. \tag{8}$$

- Via cramer's rule on the above matrix, one can find all the effects of prices and income and consumption under utility maximization. It can also show the decomposition of $\frac{dx_1}{dp_1}$ into income and substitution effects (Slutsky Equations).

7.2 Labor Supply

- Definitions: $c \equiv$ consumption, $l \equiv$ leisure, $p \equiv$ price of consumption, and $w \equiv$ wage rate.

- An individual's utility is derived from consumption and leisure, $u(c,l)$. T is the total time available to the consumer.

- The consumer chooses consumption and leisure to maximize $u(c,l)$ subject to

$$pc = (T-l)w.$$

Or,

$$\underbrace{pc + wl}_{Expenditure} = \underbrace{wT}_{Income}.$$

- Solving, we obtain the demand for consumption: $c(p,w,wT)$ and the demand for leisure: $l(p,w,wT)$. Differentiating the leisure equation with respect to wage, one obtains,

$$\frac{dl}{dw} = \frac{\partial l}{\partial w} + \frac{\partial l}{\partial wT}\frac{d(wT)}{dw} = \frac{\partial l}{\partial w} + \frac{\partial l}{\partial wT}T = \frac{\partial l}{\partial w} + \frac{\partial l}{\partial m}T.$$

Because $wT = m$, or the total possible income available if the individual worked as much as possible.

- Now the change in leisure (or $T-l=$ work hours) from a change in the wage rate can be decomposed into an income and substitution effect. Thus,

$$\frac{dl}{dw} = \underbrace{\delta^s}_{Substitution} - \underbrace{l\frac{\partial l}{\partial m}}_{Income} + \frac{\partial l}{\partial m}T.$$

Rearranging,

$$\frac{dl}{dw} = \overset{-}{\delta^s} + (T-l)\overset{+}{\frac{\partial l}{\partial m}}.$$

- If $\delta^s < (T-l)\frac{\partial l}{\partial m}$, then $\frac{dl}{dw} > 0$ so as the wage rises, people work less. Hence if the substitution effect is outweighed by a strong income effect, we see the possibility of backward bending labor supply curves.

7.3 Welfare

- Consider money as the standard for welfare in an economy. Consider the following definitions: p^0 is the initial price vector, p^1 is the final price vector, u^0 is the utility of the consumer at p^0 and u^1 is the utility of the consumer at p^1.

- Consider an initial set up of 2 commodities with price vector and corresponding utility level, p^0 and u^0. Suppose the price of good 1 falls pivoting the budget constraint out. New utility level at p^1, u^1. Instead of the price changing to reach u^1, we could have also obtained it via an increase in income. By definition, the income needed to reach u^1 at the old prices is the total expenditure: $e(p^0, u^1)$. Thus the change in income necessary for this increase in utility is:

$$e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - m \equiv \text{Equivalent Variation} \equiv \text{EV}.$$

  Note: Old Prices, New Utility. **[G-7.1]**

- Next consider an initial set up of 2 commodities with price vector and corresponding utility level, p^0 and u^0. Suppose the price of good 1 falls pivoting the budget constraint out. New utility level at p^1, u^1. After the price change, we could take away some of the consumer's income to bring him back to u^0. By definition, the income needed to reach u^0 at the new prices is the total expenditure: $e(p^1, u^0)$. Thus the change in income necessary to maintain the original level of utility is:

$$e(p^1, u^1) - e(p^1, u^0) = m - e(p^1, u^0) \equiv \text{Compensating Variation} \equiv \text{CV}.$$

  Note: New Prices, Old Utility.

- Thus this can be summarized by:

$$V(p^1, m - CV) = u^0.$$
$$V(p^0, m + EV) = u^1.$$

  In general, $CV \neq EV$.

- Suppose in the new price vector, only the price of good 1 falls, while the rest of the commodity prices remain constant. Thus $p_1^1 < p_1^0$, where the superscript represents the price vector and the subscript is the commodity index. We have defined,

$$EV = e(p^0, u^1) - m = e(p^0, u^1) - e(p^1, u^1).$$

  NOTE: $f(x^0) - f(x^1) = \int_{x^1}^{x^0} \frac{df}{dy} dy$. Thus,

$$EV = \int_{p_1^1}^{p_1^0} \frac{de}{dp} dp_1.$$

  The derivative of the expenditure function is the hicksian demand, so:

$$EV = \int_{p_1^1}^{p_1^0} h_1(p_1, p_2^0, p_3^0, ..., p_n^0, u^1) dp_1.$$

  Similarly,

$$CV = m - e(p^1, u^0) = e(p^0, u^0) - e(p^1, u^0).$$
$$CV = \int_{p_1^1}^{p_1^0} h_1(p_1, p_2^0, p_3^0, ..., p_n^0, u^0) dp_1.$$

- Note that the only difference in the CV and EV is the utility levels. If p_1 falls, $u_1 > u_0$, but comparing EV and CV will depend on if x_1 is normal or inferior.

- Consider the marshallian demand which by an identity,

$$x_1(p, m) = h_1(p, v(p, m)).$$

Since $v(p, m)$ is increasing in m, if good 1 is a normal good, $\dfrac{dx_1}{dm} > 0$. Thus $\dfrac{dh_1}{dv(p,m)} > 0$ or $\dfrac{dh_1}{du} > 0$. Thus we have $EV \geq CV$ iff the good is normal.

- Since we have defined EV and CV in terms of integrals, they can be viewed graphically as areas under curves. Consider good 1 as a normal good. Thus,

$$h_1(p_1, p_2^0, p_3^0, ..., p_n^0, u^0) \leq \underbrace{h_1\Big(p_1, p_2^0, p_3^0, ..., p_n^0, V(p_1, p_2^0, p_3^0, ..., p_n^0, m)\Big)}_{Marshallian\ Demand\ for\ x_1} \leq h_1(p_1, p_2^0, p_3^0, ..., p_n^0, u^1).$$

for $p_1^1 \leq p_1 \leq p_1^0$.

- A graphical display of this can be seen in notes. **[G-7.2]** In general, $CV \leq$ Consumer Surplus (CS) $\leq EV$.

- If Marshallian does not depend on income, $h_1(p) = x_1(p)$ or the above inequality becomes equality and $CV = CS = EV$.

- Example of Welfare comparison. Suppose there are K consumers. Let $x_i^j(p, m^j)$ be the demand for good i by consumer j having income m^j. Thus in aggregate,

$$D_i = \sum_{j=1}^{K} x_i^j(p, m^j) \equiv \text{Aggregate Demand.}$$

- More specifically, let

$$x_i^j = \alpha^j(p) + \beta(p)m^j.$$

Thus, in aggregate,

$$D_i = \sum_{j=1}^{K} \alpha^j(p) + \beta(p) \sum_{j=1}^{K} m^j.$$

- So demand for good i can be throught of as consumer j's demand with aggregate income $\sum_j m^j$. However, aggregate demands are "distribution independent" iff every consumer j has an indirect utility function,

$$v^j(p, m^j) = a^j(p) + b(p)m^j.$$

(Or any monotonic tranformation there of.)

- This type of indirect utility function is called a "Gorman Form." It is satisfied if consumers have identical homothetic preferences or quasilinear preferences (relatively strict conditions).

- Now suppose that consumers do have indirect utilities of the Gorman Form. Thus the aggregate indirect utility is,

$$V(p, m) = a(p) + b(p) + m,$$

  where $a(p) = \sum_{j=1}^{K} a^j(p)$ and $m = \sum_{j=1}^{K} m^j$.

- Suppose $V(p^1, m) - V(p^0, m) = c > 0$. At the price level p^1, the aggregate indirect utility function of this society is higher. So is society better off? Yes, but only if there is a redistribution of income. Consider new income levels:

$$(\hat{m}_1, \hat{m}^2, ..., \hat{m}^K)$$

  such that,

$$a^j(p') + b(p')\hat{m}^j = a^j(p^0) + b(p^0)m^j + \frac{c}{k}.$$

  The last term is the redistribution term of the utility surplus spread out equally over all consumers.

- Does this work? To see that it does, sum both sides across consumers,

$$V(p', \sum_j \hat{m}^j) = V(p^0, \sum_j m^j) + c.$$

  Hence,

$$\sum_{j=1}^{K} \hat{m}^j = m.$$

  Moreover, at income levels, $(\hat{m}_1, \hat{m}^2, ..., \hat{m}^K)$, every consumer is better off. Fascinating.

8 Week 8

8.1 Cost of Living Indices

- Suppose p^0 is the initial price vector and corresponds to a utility function for the consumer of u^0. Also, p^1 is the final price vector corresponding to the utility for the consumer of u^1.

- A true measure of the cost of living would be the ratio of the expenditure functions at the different prices levels that would result in the same utility level:

$$\frac{e(p^1, u^0)}{e(p^0, u^0)}.$$

- But since utility levels are unobservable, we estimate it with the Laspeyres index, L, such that,

$$L = \frac{P^1 X^0}{P^0 X^0},$$

  where X^0 is the initial bundle of goods.

- In all likelihood, $L \geq \dfrac{e(p^1, u^0)}{e(p^0, u^0)}$ because the L index compares the same bundle of goods. When the price level changes, consumers will reoptimize their consumption and most likely they will consume different quantities of goods in the bundle depending on the new relative prices. Thus, when there is a price decrease in the economy for example, the L estimate of the cost living over estimates the true measure.

8.2 Markets

- We have developed a measure for the concept of welfare for the consumer: EV, CV, and consumer surplus. But now we would like to find a similar (monetary) measure for the producer. The obvious candidate is profit.

- Suppose the output price increases from p^0 to p^1 where these prices are single prices and not a vector as in previous sections.

- The change in profits for the producer is:

$$\Delta = \pi(p^1) - \pi(p^0).$$

- Via the hotelling Lemma,

$$\frac{\partial \pi(p, w)}{\partial p} = y(p, w).$$

  Thus, we can write Δ as an integral as follows:

$$\Delta = \int_{p^0}^{p^1} \frac{\partial \pi(p, w)}{\partial p} dp = \int_{p^0}^{p^1} y(p) dp.$$

- Therefore, as seen in the notes [**G-8.1**], the difference in profits, Δ, or otherwise called "Producer Surplus," can be shown as the different between the two prices under the supply curve.

- The same graph can be drawn for aggregate supply with the area representing aggregate producer surplus.

8.3 Price Formation

- Equilibrium market price is defined in elementary economics where Demand, $D(p)$, equals Supply, $S(p)$.

- To measure the responsiveness of these curves to prices, one could look at the slopes of the curves, but unfortunately because of a difference in units, two equilivalent demand or supply functions might have different slopes. Thus, consider elasticities:

$$\text{Elasticity of Demand: } \varepsilon^D = \frac{dD(p)}{dp}\frac{p}{D(p)}.$$

$$\text{Elasticity of Supply: } \varepsilon^S = \frac{dS(p)}{dp}\frac{p}{S(p)}.$$

8.4 Taxes

- Two types of taxes, quantity taxes and value taxes.

- Let p^D be the price paid by the consumer and let p^S be the price received by producers.

- A Quantity tax at rate t per unit would be of form:

$$p^D = p^S + t.$$

- A Value tax at a rate t would be of the form:

$$p^D = p^S(1+t).$$

- Graphically it can be shown (See notes for graph [**G-8.2**]) that the tax revenue from the tax is less than the loss of consumer and produce surpluses together. Thus, taxes are distortionary and inefficient and result in a dead weight loss:

$$\text{Dead Weight Loss} = \text{Consumer Surplus} + \text{Producer Surplus - Tax Revenue} > 0.$$

8.5 General Equilibrium - Beyond Mock Exam

- Consider a pure exchange economy with NO production.
- Suppose there are n commodities and K consumers who wish to trade their endowments of goods.
- Each consumer J is described as hairy and ugly (Ed) with (u^J, w^J) where u^J is J's utility function and w^J is J's endowment such that,

$$w^J = (w_1^J, w_2^J, \ldots, w_n^J).$$

- Consider the following example which will be developed throughout the analysis.
  - Suppose $K = 2$ and $n = 2$. Let $w^1 = (3,1)$ and $w^2 = (1,3)$. The utility functions of the two consumers are cobb-douglas such that,

    $$u^1 = x_1^1 x_2^1,$$

    and,

    $$u^2 = x_1^2 x_2^2.$$

    Note that the superscripts are consumer indices and the subscripts are commodity indices.
- An allocation, $x = (x^1, x^2, \ldots, x^K)$ assigns a consumption bundle $x^J = (x_1^J, x_2^J, \ldots, x_K^J)$ to each consumer $J = 1...K$.
- An allocation, $x = (x^1, x^2, \ldots, x^K)$ is feasible if:

$$\sum_{J=1}^{K} x^J \leq \sum_{J=1}^{K} w^j.$$

  This just means that an allocation is feasible if the quantities consumed in the bundle do not exceed the total endowments of the goods.

  - In our example, $x^1 = x^2 = (2,2)$ is a feasible allocation. Note that allocations that add up to less than the total endowment are obviously feasible as well even though it would result in having to destroy some goods to reach that allocation.

8.6 Walrasian Equilibrium

- Walrasian equilibrium, or otherwise called competitive or normal equilibrium, is a pair $(\bar{p}, \bar{x})$ where,

$$\bar{p} = (\bar{p}_1, \ldots, \bar{p}_n)$$

  is a price vector and,

$$\bar{x} = (\bar{x}^1, \ldots, \bar{x}^K)$$

  is an allocation such that,

- (1) $\bar{x}$ is feasible.
- (2) $\bar{x}^J$ maximizes $u^J(x^J)$ such that $\bar{p}x^J \leq \bar{p}w^J$.

- So the allocation is feasible in that the total allocation quantities do not exceed the total endowments. Also the allocation maximizes the consumer utility such that at the current price level, each consumer's bundle is affordable for him.

- Back to our example.
  - The Marshallian demands coming from the cobb-douglas utility functions are:
$$x_1^1 = \frac{3p_1 + p_2}{2p_1} \text{ and } x_2^1 = \frac{3p_1 + p_2}{2p_2}.$$
$$x_1^2 = \frac{p_1 + 3p_2}{2p_1} \text{ and } x_2^2 = \frac{p_1 + 3p_2}{2p_2}.$$
  - Looking at the individual market conditions (for each good):
$$\text{Good 1: } \frac{3p_1 + p_2}{2p_1} + \frac{p_1 + 3p_2}{2p_1} = 4.$$
$$\text{Good 2: } \frac{3p_1 + p_2}{2p_1} + \frac{p_1 + 3p_2}{2p_1} = 4.$$
  - Solving the good 1 equation yields,
$$p_1 = p_2.$$
  - Substituting this into the market for good 2 just tells us that the equation is valid! We cannot actually solve for either prices but we do find that the prices must be equal. Thus an equilibrium supports any set of prices such that the two prices are equal.
  - NOTE: Once we found the equilibrium in market 1, this automatically implied equilibrium in market 2.

- Back in the general setting, consider the vector of demands for consumer J:
$$x^J(p, pw^J).$$

- $x^J(p, pw^J)$ is homogeneous of degree 0 in prices because if you increase prices by t, you increase both income (via the endowments) and the price of goods the consumer purchases. The net effect is 0. Thus, if $\bar{p}$ is an equilibrium price vector, so is $t\bar{p}$. (Thus we have a multiplicity of equilibrium price vectors.)

- Define a vector of excess demands, Z^J, as:
$$Z^J(p, pw^J) = x^J(p, pw^J) - w^J.$$
Z^J is how much more of each good in the bundle that consumer J wants beyond his endowment. Clearly Z_i^J could be negative if a consumer would rather sell his endowment than have more of good i.

- Assuming local non-satiation, the value of excess demand must be 0 because consumers choose bundles on their budget constraint. Thus,

$$pZ^J(p, pw^J) = 0.$$

- In the aggregate, define:

$$Z(p) = \sum_{J=1}^{K} Z^J(p, pw^J),$$

as the aggregate excess demand over all consumers.

- Since $pZ^J(p, pw^J) = 0$ for all $J = 1...K$ consumers, then,

$$pZ(p) = 0.$$

- Thus Walras' Law:

"The value of aggregate excess demand must be zero."

Which in other words just means that aggregate expenditure equals aggregate income.

- Consider the following implication of Walras' Law:
  - Walras' law says:

  $$pZ(P) = p_1 Z_1(p) + p_2 Z_2(p) + \cdots + p_n Z_n(p) = 0.$$

  Suppose we have equilibrium in all but the n^{th} market. Thus

  $$p_i Z_i(p) = 0 \; \forall \; i < n.$$

  Thus, the last market must also be in equilibrium. So $p_n Z_n(p) = 0$.

- NOTE: When we solve for competitive equilibrium prices, we set one price equal to one (ie $p_1 = 1$) or define a numeraire. All other prices are measured in terms of the numeraire.

8.7 Pareto Efficiency

- Consider K consumers. An allocation, $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^K)$ is pareto efficient if the following is true:
  - (1) $\bar{x}$ is feasible.
  - (2) $\nexists$ a different feasible allocation, $\tilde{x} = (\tilde{x}^1, \ldots, \tilde{x}^K) \ni$

  $$u^J(\tilde{x}^J) \geq u^J(\bar{x}^J) \; \forall \; J = 1...K,$$

  and,

  $$u^J(\tilde{x}^J) > u^J(\bar{x}^J) \text{ for some } J.$$

- Thus all consumers cannot be at least as good off and at least one consumer cannot be strictly better off.

9 Week 9

9.1 The First Theorem of Welfare

- Assume $L.N.S.$ Then any competitive equilibrium allocation is pareto efficient.
  Proof: Consider a competitive equilibrium, $(\bar{p}, \bar{x})$. Proceed by contradiction. Suppose $\bar{x}$ is not pareto efficient. There there exists a different feasible allocation $\hat{x}$, such that,

$$(1): \quad u^J(\hat{x}^J) \geq u^J(\bar{x}^J), J = 1...K.$$

$$(2): \quad u^J(\hat{x}^J) > u^J(\bar{x}^J), \text{ for at least one } J.$$

  By LNS, (1) implies, $\bar{p}\hat{x}^J \geq \bar{p}w^J$. If not, there would exist another bundle, $\tilde{x}^J$, that would make the consumer strictly better off: $u^J(\tilde{x}^J) > u^J(\hat{x}^J) \geq u^J(\bar{x}^J)$.
  Therefore, $\bar{p}\hat{x}^J \geq \bar{p}w^J$ for $J = 1...K$. Hence (2) implies that at the equilibrium prices, the consumer cannot afford $\hat{x}^J(p\hat{x}^J > pw^J)$. Then summing across consumers,

$$\bar{p}\sum_{J=1}^{K}\hat{x}^J > \bar{p}\sum_{J=1}^{K}w^J.$$

  Which contradicts feasibility. QED.

- Note that pareto efficiency does not imply desirability.

9.2 Is free trade a good thing?

- Suppose, utilities of consumers 1 and 2 with superscripts representing consumer indices and the subscripts representing commodity indices:

$$u^1 = x_1^1 x_2^1$$

  and

$$u^2 = x_1^2 x_2^2$$

  with endowments:

$$w^1 = (3, 1)$$

  and

$$w^2 = (1, 3).$$

- The equilibrium price would therefore be $p_1 = p_2 = 1$ and the equilibrium allocation would be $(2, 2)$ and $(2, 2)$.

- Suppose now the utility of consumer 1 is:

$$u^1 = x_1^1 x_2^1 - 2x_1^2.$$

  And for consumer 2:

$$u^2 = x_1^2 x_2^2 - 2x_2^1.$$

- Thus both consumers experience negative externalities from the other's consumption. Plugging in the equilibrium values $(2,2)$,$(2,2)$ gives us $u^1 = u^2 = 0$. However, if trade is not allowed and we consider the endowment utility level $(3,1),(1,3)$, the utilities are $u^1 = u^2 = 1$. Thus the consumers are better off without trade! Thus a reason for regulation.

9.3 Edgeworth Box Analysis

- Refer to graphs in notes **[G-9.1]**. Under this setting the social planner's problem is: Choose an allocation (x^1, x^2) to maximize,

$$u^1(x^1)$$

subject to:

$$u^2(x^2) - \bar{u}.$$

- The first order conditions are therefore tangency of each consumer's indifference curves to each other (and of course to their budget constraints.) Thus,

$$MRS^1 = MRS^2.$$

And,

$$x^1 + x^2 = w^1 + w^2.$$

The second contraint is just the feasibility constraint.

- Note that this condition only holds for interior solutions.
- The set of Pareto efficient allocations is called the Contract Curve. **[G-9.2]**
- The Core is the area on the contract curve that makes both consumers better off (hence it is in "DEE LENZ.")
- Since there are many pareto efficient equilibrium, which one is optimal? Thus there is room for regulation, but not in setting consumption levels but rather in setting endowments and letting the market determine where people consume (Hence the Second Theorem of Welfare).
- The FOC's mentioned above for determining pareto efficient equilibria relies on the fact that preferences are convex.

9.4 Competitive Equilibrium with Production

- Consider 2 production processes that produce 2 commodities, y_1 and y_2 using the same input, x.
- Production process 1 is owned by agent A and is defined as:

$$y_1 = \sqrt{x_1}.$$

- Production process 2 is owned by agent B and is defined as:

$$y_2 = \sqrt{x_2}.$$

- The two agents, A and B, also have utility functions,

$$u^A = y_1^A y_2^A.$$
$$u^B = y_1^B y_2^B.$$

- Consider intial endowments,

$$w^A = (y_1, y_2, x) = (0, 0, 5).$$
$$w^B = (y_1, y_2, x) = (0, 0, 3).$$

- So the total amount of X available in the economy is 8 and note that the two agents get no utility from possessing x so they would rather use it in production.

- Define P_1 as the price of y_1 and P_2 as the price of y_2. Let r be the price of x. Set a numeraire,

$$P_2 = 1.$$

  Thus the price ratio is defined as,

$$P = \frac{P_1}{P_2} = \frac{P_1}{1} = P_1.$$

- As producers, both A and B maximize profits. Define profits:

$$\pi^A = P_1 y_1 - r x_1 = P_1 \sqrt{x_1} - r x_1.$$
$$\pi^B = P_2 y_2 - r x_2 = P_2 \sqrt{x_2} - r x_2.$$

- Thus first order conditions,

$$\frac{\partial \pi^A}{\partial x_1} \Rightarrow \frac{1}{2} P_1 x_1^{-1/2} - r = \frac{1}{2} P x_1^{-1/2} - r = 0.$$
$$\frac{\partial \pi^B}{\partial x_2} \Rightarrow \frac{1}{2} P_2 x_2^{-1/2} - r = \frac{1}{2} x_2^{-1/2} - r = 0.$$

- Solving the FOC's for x_1:

$$\frac{1}{2} P x_1^{-1/2} - r = 0.$$
$$P x_1^{-1/2} = 2r.$$
$$x_1^{-1/2} = \frac{2r}{P}.$$
$$x_1^{1/2} = \frac{P}{2r}.$$
$$x_1 = \frac{P^2}{4r^2}.$$

- Solving the FOC's for x_2:

$$\frac{1}{2}x_2^{-1/2} - r = 0.$$
$$x_2^{-1/2} = 2r.$$
$$x_2^{1/2} = \frac{1}{2r}.$$
$$x_2 = \frac{1}{4r^2}.$$

- Substituting the input demand into the production function yields the supply of y_1 and y_2,

$$y_1 = \sqrt{x_1} = \sqrt{\frac{P^2}{4r^2}} = \frac{P}{2r}.$$
$$y_2 = \sqrt{x_2} = \sqrt{\frac{1}{4r^2}} = \frac{1}{2r}.$$

Therefore substituting into the profit functions,

$$\pi^A = Py_1 - rx_1 = P\frac{P}{2r} - r\frac{P^2}{4r^2} = \frac{2P^2}{4r} - \frac{P^2}{4r} = \frac{P^2}{4r}.$$
$$\pi^B = y_2 - rx_2 = \frac{1}{2r} - r\frac{1}{4r^2} = \frac{2}{4r} - \frac{1}{4r} = \frac{1}{4r}.$$

- At the same time, both consumers maximize their utility. The income of each consumer is made up of their profits from making y_i, and also what they get from selling there endowment of x. Thus,

$$m^A = \frac{P^2}{4r} + 5r.$$

And,

$$m^B = \frac{1}{4r} + 3r.$$

- Because the utility functions are cobb-douglas, we can write the quantity demands of y_i as each consumer's income level divided by two times the prices level. The demand for commodity y_1 for example:

$$y_1^A = \frac{\frac{P^2}{4r} + 5r}{2P}.$$
$$y_1^B = \frac{\frac{1}{4r} + 3r}{2P}.$$

Similiarly,

$$y_2^A = \frac{\frac{P^2}{4r} + 5r}{2}.$$

$$y_2^B = \frac{\frac{1}{4r} + 3r}{2}.$$

- Setting supply of y_1 equal to the demand for y_1,

$$\frac{\frac{P^2}{4r} + 5r}{2P} + \frac{\frac{1}{4r} + 3r}{2P} = \frac{P}{2r}.$$

$$\frac{1}{2P}\left[\frac{P^2}{4r} + 5r + \frac{1}{4r} + 3r\right] = \frac{P}{2r}.$$

$$\frac{P^2}{2} + 10r^2 + \frac{1}{2} + 6r^2 = 2P^2.$$

$$\frac{P^2}{2} + 16r^2 + \frac{1}{2} = 2P^2.$$

$$P^2 + 32r^2 + 1 = 4P^2.$$

$$32r^2 + 1 = 3P^2.$$

We also have the condition that the supply of x must add up to the total endowment of x. Thus,

$$\frac{P^2}{4r^2} + \frac{1}{4r^2} = 8.$$

Thus,

$$P^2 + 1 = 32r^2.$$

Substituting this into the equation above,

$$(P^2 + 1) + 1 = 3P^2.$$

$$P^2 + 2 = 3P^2.$$

$$2 = 2P^2.$$

$$P = 1.$$

Substituting this into the above equation to solve for r,

$$32r^2 + 1 = 3(1).$$

$$32r^2 = 2.$$

$$r^2 = \frac{1}{16} \Rightarrow r = \frac{1}{4}.$$

Substituting in again to find the quantity of y_1^A and y_2^B,

$$y_1^A = \frac{\frac{P^2}{4r} + 5r}{2P} = \frac{\frac{1}{1} + \frac{5}{4}}{2} = \frac{9}{8}.$$

$$y_1^B = \frac{\frac{1}{4r} + 3r}{2P} = \frac{\frac{1}{1} + \frac{3}{4}}{2} = \frac{7}{8}.$$

- Setting supply of y_2 equal to the demand for y_2,

$$\frac{\frac{P^2}{4r} + 5r}{2} + \frac{\frac{1}{4r} + 3r}{2} = \frac{1}{2r}.$$

But we don't need to solve this system because we used the numeraire so we can just plug in P and r to find the quantity of y_2^A and y_2^B:

$$y_2^A = \frac{\frac{P^2}{4r} + 5r}{2} = \frac{\frac{1}{1} + \frac{5}{4}}{2} = \frac{9}{8}.$$

$$y_2^B = \frac{\frac{1}{4r} + 3r}{2} == \frac{\frac{1}{1} + \frac{3}{4}}{2} = \frac{7}{8}.$$

9.5 Two Production Processes - Marginal Rate of Transformation

- Consider two production processes: $y_1 = f_1(x_1)$ and $y_2 = f_2(x_2)$ where $x_1 + x_2 = x$. We can define a production possibilities frontier (see graph in notes [**G-9.3**]) such that,

$$y_2 = T(y_1).$$

- Thus, define the Marginal Rate of Transformation (MRT) as the slope of this curve:

$$\frac{dT(y_1)}{dy_1} = -\frac{f_2'}{f_1'} = -\frac{\partial y_2/\partial x_2}{\partial y_1/\partial x_1}.$$

- The MRT is the trade off between y_1 and y_2 given that the input, x, is shared between the two production processes.

10 Week 10

10.1 More on Pareto Efficiency

- With reference to the model from last week, we had two agents acting as both producers and consumers. We get pareto efficiency in the following way. Choose $(y_1^A, y_1^B, y_2^A, y_2^B)$ to maximize,
$$u^A(y_1^A, y_2^A),$$
subject to,
$$u^B(y_1^B, y_2^B) = \bar{u},$$
and,
$$y_2^A + y_2^B = T(y_1^A + y_1^B).$$
- So we maximize the one agents utility subject to the other attaining some fixed level of utility. The second constraint just provides that we can get the full amount of y_2 by applying technology to y_1.
- By varying $\bar{u}$, we can find all possible pareto efficient points.
- Setting up the Lagrangian of the maximation above, yeilds FOCs. Solving the FOC's yields,
$$MRS^A = MRS^B = MRT.$$

10.2 Decision Under Uncertainty

- Let C be the set of N possible outcomes.
- A simply lottery, L, is a probability distribution, $(P_1, \ldots, P_N)$ over C, where P_i is the probability of outcome i.
- Example $C = (0, 100)$ and $L = (\frac{1}{2}, \frac{1}{2})$ is like a simple coin toss lottery with equal probability of winning 0 or 100.
- A Compound Lottery is $(\alpha^1, \ldots, \alpha^K, L^1, \ldots, L^K)$ where α^J is the probability of lottery L^J. From this compound lottery, we get a simple lottery, $(P_1, \ldots, P_N)$ where,
$$P_i = \sum_{J=1}^{K} \alpha^J P_i^J,$$
where P_i^J is the probability of outcome i in lottery J.
- Example of a Compound Lottery. Let $C = (0, 100)$, $L^1 = (\frac{1}{2}, \frac{1}{2})$, and $L^2 = (0, 1)$. Thus the compound lottery is $(\frac{1}{2}, \frac{1}{2}, L^1, L^2)$. The outcomes are still 0 and 100. But now consider the probabilities of each of these outcomes. They are simply equal to

the probability of being in each lottery times the probability of attaining the outcome. Thus,

$$Prob(0) = \alpha^1 P_1^1 + \alpha^2 P_1^2 = \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * 0 = \frac{1}{4}.$$

$$Prob(100) = \alpha^1 P_2^1 + \alpha^2 P_2^2 = \frac{1}{2} * \frac{1}{2} + \frac{1}{2} * 1 = \frac{3}{4}.$$

Thus the simple lottery becomes $(\frac{1}{4}, \frac{3}{4})$.

10.3 Preference Properties Involving Lotteries

- Let $\mathbb{L}$ be the set of simple lotteries and let $\succeq^P$ represent preferences of $\mathbb{L}$. We have several assumptions about $\succeq^P$.

- Assumption 1: $\succeq^P$ is complete and transitive.

- Assumption 2: Continuity. For any 3 simple lotteries, L, L', and $L'' \in \mathbb{L}$, the sets,

$$\left\{ \mathbb{L} \in [0,1] \ni \alpha L + (1-\alpha)L' \succeq^P L'' \right\}$$

  and,

$$\left\{ \mathbb{L} \in [0,1] \ni L'' \succeq^P \alpha L + (1-\alpha)L' \right\}$$

  are closed sets. This is just a technical assumption. It says we have no jumps in preferences.

- Assumption 3: Independence Axiom. For an 3 lotteries, L, L', and L'', and $\alpha \in [0,1]$, then,

$$L \succeq^P L' \text{ if and only if } \alpha L + (1-\alpha)L'' \succeq^P \alpha L' + (1-\alpha)L''.$$

  Note that L'' appears on both sides, so we really shouldn't take it into account when determining our preferences between L and L'.

- Note that the independence axiom assumes the independence of the outcomes, but clearly in this case, you wouldn't want to read about Paris knowing that you weren't ever going to get to go. People have regrets. But I still think this was a poor example.

- Now assuming that the assumptions and axioms above hold, then there exists a vector of utilities $(u_1, \ldots, u_N) \ni$ for any 2 lotteries, $L = (P_1, \ldots, P_N)$ and $L' = (P_1', \ldots, P_N')$,

$$L \succeq^P L' \text{ if and only if } EU(L) \geq EU(L').$$

$$L \succeq^P L' \text{ if and only if } \sum_{i=1}^{N} P_i u_i \geq \sum_{i=1}^{N} P_i' u_i.$$

  Where $EU(\cdot)$ is the expected utility of the lottery.

- Note that if $(\bar{u}_1, \ldots, \bar{u}_N)$ represents the same preferences as $(u_1, \ldots, u_N)$, then it must be the case that,

$$\bar{u}_i = \beta u_i + \gamma,$$

with $\beta > 0$ and γ a constant. Note that just having a monotonic transformation of a utility function is not enough (not strong enough). The transformation must be linear and increasing to maintain the preference relationships.

- Consider the importance of the Independence axiom with the following example. Let $C = (C_1, C_2, C_3)$, $L = (1, 0, 0)$, $L' = (0, 1, 0)$, and $L'' = (0, 0, 1)$. Thus,

$$\alpha L + (1 - \alpha)L'' \text{ has utility: } \alpha u_1 + (1 - \alpha)u_3.$$
$$\alpha L' + (1 - \alpha)L'' \text{ has utility: } \alpha u_2 + (1 - \alpha)u_3.$$

Now suppose,

$$L \succeq^P L'.$$

Therefore by the independence axiom,

$$\alpha L + (1 - \alpha)L'' \succeq^P \alpha L' + (1 - \alpha)L''.$$

Thus from above,

$$\alpha u_1 + (1 - \alpha)u_3 \geq \alpha u_2 + (1 - \alpha)u_3.$$

Simplifying,

$$u_1 \geq u_2.$$

Fascinating.

10.4 Risk Preferences

- Now consider outcomes as "wealth" or "consumption." A lottery in this case is a cummulative distribution function, $F(x)$ where $F(x) = \text{Prob}(\text{"wealth"} \leq x)$.

- The utility of $F(x)$ is:

$$\int u(x)dF(x).$$

- If $f(x)$ is the density of F, the utility is then,

$$\int u(x)f(x)dx.$$

Which is the definition of expected utility as each level of utility, $u(x)$, is weighted by its probability, $f(x)$. In this case $u(x)$ is called a "Bernoulli" or "Von Neumann - Morganstern" (VNM) utility function.

- 3 cases.
  - For any lottery, $F(x)$, then

$$u\left(\int xdF(x)\right) \geq \int u(x)dF(x)$$

implies Risk Aversion and Concave Utility Functions.

  - For any lottery, $F(x)$, then

$$u\left(\int xdF(x)\right) = \int u(x)dF(x)$$

implies Risk Neutrality and Linear Utility Functions.

  - For any lottery, $F(x)$, then

$$u\left(\int xdF(x)\right) \leq \int u(x)dF(x)$$

implies Risk Lovingness and Convex Utility Functions.

10.5 Example of the Demand for Insurance

- Let M = initial wealth. L = Possible Loss. P = Probability of Loss. S = Amount of Insurance Coverage. r = unit premium or price of insurance.
- Suppose the consumer buys S units of insurance. Then the lottery is as follows: With probability $(1-P)$, the consumer does not experience a loss and has payoff $M-rS$. With probablity, P, the consumer has the loss and has payoff $M-L-rS+S = M-L+(1-r)S$.
- The consumer chooses S to maximize,

$$(1-p)u\left[M-rS\right] + pu\left[M-L+(1-r)S\right].$$

Yields FOC(S):

$$-(1-p)ru'(M-rS) + p(1-r)u'(M-L+(1-r)S) = 0.$$

- Now consider the case of “Fair Insurance,” or $p = r$. The first order condition becomes,

$$(1-p)ru'(M-rS) = p(1-r)u'(M-L+(1-r)S).$$

$$u'(M-rS) = u'(M-L+(1-r)S).$$
$$M-rS = M-L+(1-r)S.$$
$$-rS = -L+(1-r)S.$$
$$L = S.$$

  Thus ”Fair Insurance” implies ”Full Insurance.”

- NOTE!!!! that the *SOC* for maximization is only satisfied if the consumer if risk averse. Otherwise it IS NOT! Important!!

- Finally, since we have shown that two utility functions representing the same preferences can have different second order derivatives, (as one is a linear combination of the other), we cannot rely on $u''(x)$ alone to determine the concavity or convexity of preferences and therefore the risk level of the consumer.

- Define: The Coefficient of Absolute Risk Aversion (CARA) as,

$$r(x) = -\frac{u''(x)}{u'(x)}.$$

- Example: $u(x) = -e^{-ax}$. Thus $u'(x) = ae^{-ax}$ and $u''(x) = -a^2e^{-ax}$. Thus,

$$r(x) = -\frac{u''(x)}{u'(x)} = -\frac{-a^2e^{-ax}}{ae^{-ax}} = a.$$

- Suppose that $u^1(x)$ has a higher CARA than $u^2(x)$. Then $u^1(x) = \phi u^2(x)$ where ϕ is concave. Thus $u^1(x)$ is more concave and represents a higher degree of risk aversion. So the higher the value of CARA that a consumer’s utility function has, the more concave it is and thus the more risk averse the consumer is.

- Note that subjectivity plays a role in determining probablities of all these types of problems.