Microeconomics I
Lent Term

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1 Week 1: 14 Jan - 18 Jan

1.1 Introduction to John Sutton's Lectures

- Last term we studied perfect competition type models that relied on three assumptions: Large number of firms, Homogeneous product, and Full Information. Having these 3 assumptions allowed us to treat each firm as a price taker. We can then characterise equilibrium via supply=demand which is the heart of "Walrasian Equilibrium." This term we will move beyond the competitive equilibrium and relax the assumptions one by one. This will lead directly to the notion of a Nash Equilibrium.

- This Term's Agenda.
  - 1) Suppose a small number of firms: monopoly and oligopoly.
  - 2) Suppose heterogeneous products.
  - 3) Suppose there is incomplete information in the market for either the consumers or producers.
  - 4) General Equilibrium, Trade and Welfare under perfect information and under imperfect information.

1.2 Part I: The Monopoly Model

- Review of elementary ideas. The Monopolist's problem is to maximize profit such that,

$$\pi = pq - C(q) = R(q) - C(q).$$

Where $R(q) = pq \equiv$ a revenue function. The optimal solution take several forms.

  - Solution 1:

$$\frac{d\pi}{dq} = \frac{dR}{dq} - \frac{dC}{dq} = 0.$$
$$MR = MC.$$

  - Solution 2 (explicit):

$$\frac{d\pi}{dq} = p + q\frac{dp}{dq} - \frac{dC}{dq} = 0.$$

Dividing out a p,

$$\frac{d\pi}{dq} = p(1 + \frac{q}{p}\frac{dp}{dq}) - \frac{dC}{dq} = 0.$$

Inverting,

$$\frac{d\pi}{dq} = p(1 + \frac{1}{\frac{p}{q}\frac{dq}{dp}}) - \frac{dC}{dq} = 0.$$

Note that the elasticity of demand is defined as, $\eta = -\frac{p}{q}\frac{dq}{dp}$. Thus,

$$\frac{d\pi}{dq} = p(1 - \frac{1}{\eta}) - \frac{dC}{dq} = 0.$$

$$p(1 - \frac{1}{\eta}) = \frac{dC}{dq}.$$

$$p(1 - \frac{1}{\eta}) = MC.$$

- Some notes to mention about the second form of the solution:
  - 1) Since $\eta < \infty$, $p > MC$ (as in general, the demand curve is downward sloping).
  - 2) The optimum always occurs at a point where $\eta > 1$. This is clear because $0 < \frac{dC}{dq} = p(1 - \frac{1}{\eta})$. Hence the RHS of this expression must also be positive which means $\eta > 1$.
  - 3) Note that the divergence of price from MC is a measure of the so-called "Degree of Monopoly Power." Also, rewriting the FOC,

    $$\underbrace{\frac{p - MC}{p}}_{Lerner\ Index} = \frac{1}{\eta}.$$

    Lerner taught at the LSE in the past.
  - 4) Note that in general, $\pi > 0$ at equilibrium and the profit can be interpreted as a rent on the "property right" that confers in monopoly. This way of thinking about monopoly profit will be useful in future analysis.
  - 5) This model is consistent with increasing returns in the sense that MC may be downward sloping. All that matters is that MR cuts MC from above. See graph. [**G-1.1**]

1.3 A Simple Illustration of Price Discrimination

- Note that we are now refering to what is known as "3rd degree price discrimination."
- A monoplist sells in a number of geographically separated markets between which arbitrage is not feasible.
- Suppose we have 2 markets with demand schedules whose elasticities are η_1 and η_2 respectively.
- The profit maximization problem takes the form:

  $$\pi = p_1 q_1 + p_2 q_2 - C(q).$$

  Notice that there is only one cost function and $q = q_1 + q_2$.

- Maximizing with respect to q_1 and following the same steps as previously,

$$\frac{d\pi}{dq_1} = p_1 + q_1\frac{dp}{dq_1} - \frac{dC}{dq} = 0.$$

$$\frac{d\pi}{dq_1} = p_1(1 + \frac{q_1}{p_1}\frac{dp}{dq_1}) - \frac{dC}{dq} = 0.$$

$$\frac{d\pi}{dq_1} = p_1(1 + \frac{1}{\frac{p_1}{q_1}\frac{dq_1}{dp_1}}) - \frac{dC}{dq} = 0.$$

$$\frac{d\pi}{dq_1} = p_1(1 - \frac{1}{\eta_1}) - \frac{dC}{dq} = 0.$$

$$MR_1 = MC.$$

- Similarly, maximizing with respect to q_2 and following the same steps are previously,

$$\frac{d\pi}{dq_2} = p_2 + q_2\frac{dp_2}{dq_2} - \frac{dC}{dq} = 0.$$

$$\frac{d\pi}{dq_2} = p_2(1 + \frac{q_2}{p_2}\frac{dp_2}{dq_2}) - \frac{dC}{dq} = 0.$$

$$\frac{d\pi}{dq_2} = p_2(1 + \frac{1}{\frac{p_2}{q_2}\frac{dq_2}{dp_2}}) - \frac{dC}{dq} = 0.$$

$$\frac{d\pi}{dq_2} = p_2(1 - \frac{1}{\eta_2}) - \frac{dC}{dq} = 0.$$

$$MR_2 = MC.$$

- Setting $MR_1 = MR_2$, we immediately get:

$$\frac{p_1}{p_2} = \frac{1 - 1/\eta_2}{1 - 1/\eta_1}.$$

  So suppose that $\eta_2 > \eta_1$. This means that the demand curve in the second market is more elastic or flatter than in the first market. Thus the prices will be higher in market 1, the market with the more inelastic demand curve. A 10 percent increase in prices in market 1 results in a less than 10 percent decrease in quantity demanded. This is seen clearly from this equation as well.

- Graphically, as shown in the notes [**G-1.2**], we can easily determine the individual market prices and quantities. Simply sum horizontally the MR curves of each of the two markets and set $\sum MR = MC$ in the overall market. This determines the equilibrium level of MR. Note, of course, $MR_1 = MR_2 = MC$. Once one determines the level of MR, go back into the individual markets and equate the quantity produced with this MR and then go up to the individual demand curves to find the prices.

1.4 A Formal Treatment of Price Discrimination

- Reference: Schmalensee.
- Consider a market with N firms and let each firm have constant marginal cost equal to c. The firm's profit function is therefore,

$$\underbrace{\pi = \sum_{i=1}^{N}(p_i - c)q_i(p_i) = \sum_{i=1}^{N}\pi_i(p_i)}_{\textit{Equation 1}}.$$

- Assume that $\pi(\cdot)$ is smooth (all derivatives exist) and strictly concave. This implies that we have a smooth decreasing MR schedule.
- Consider first the "No Discrimination" Regime. This yields FOC:

$$\underbrace{\sum_{i=1}^{N}\pi_i'(p^*) = \sum_{i=1}^{N}\Big[(p^* - c)q_i'(p^*) + q_i(p^*)\Big]}_{\textit{Equation 2}},$$

  with p^* being the optimum price in all markets. Note we are now maximizing with respect to price.
- Now consider the "Discrimination" Regime. This yields FOC:

$$\underbrace{\pi'(p_i^*) = \Big[(p_i^* - c)q_i'(p_i^*) + q_i(p_i^*)\Big]\ \forall\ i = 1\ldots N}_{\textit{Equation 3}},$$

  with p_i^* being the optimum price in market i.
- Before we go any further we need to define some preliminaries.
  - 1) Label the market as follows:
    - "Strong Market" if $p_i^* > p^*$.
    - "Weak Market" if $p_i^* < p^*$.
  - 2) Assume income effects are small.
  - 3) Ignore distributional effects.
  - 4) This justifies the use of the simple welfare indicator of profits plus consumer surplus.
  - Define the welfare indicator function as:

$$\underbrace{W = \sum_{i=1}^{N}\left\{\overbrace{\int_{p_i}^{\infty} q_i(p)dp}^{CS} + \overbrace{\pi(p_i)}^{Profit}\right\}}_{\textit{Equation 4}}.$$

- We now construct an "artificial problem" as follows:

$$max_{p_i} \;\; \pi(p_1, p_2, \ldots, p_N),$$

subject to,

$$\sum_{i=1}^{N} \pi_i'(p^*)(p_i - p^*) \leq t.$$

This may look a little strange at first but its construction will give us the power to define discrimination just by varying the level of t. If t is large, the constraint will be non-binding so we collapse to equation 3, the discrimination case. Why? Well the first term in the constraint will surely not be equal to zero because if it was, there would be no reason for discrimination. Charging everyone p^* would be profit maximizing. Thus the term that could be driven down to zero is the second term which represents how much discrimination is being invoked. So if t is large, the non-discrimination constraint is not-binding and we would expect that the solution to this problem would be the solution to equation 3, under the discrimination regime. On the other hand, if t is very small, close to zero, then the constraint is binding. Since the first term, as we said, is clearly not zero, the second term must be driven to zero. Thus the market prices approach the equilibrium price, p^* which is charged in all markets. Thus the solution would collapse to equation 2, under the non-discrimination regime.

- Define a solution to this problem as $p_i(t)$. The lagrangian is written as:

$$\mathbb{L} = \sum_{i=1}^{N} \pi(p_i) - \lambda \left[\sum_{i=1}^{N} \pi_i'(p^*)(p_i - p^*) - t \right].$$

- This yields FOC:

$$\pi_i'(p_i(t)) = \lambda \pi_i'(p^*), for\ i = 1, \ldots, N.$$

- Note that if the constraint is non-binding ($t \rightarrow \infty$), then $\lambda = 0$, so $\pi_i'(p_i(t)) = 0$.

- For large t, the constraint in the artificial problem is non-binding and the FOC coincides to the FOC for equation 3. It corresponds to the solution, p_i^*. Here $\lambda = 0$ and we can write,

$$p_i^*(t) = p_i^*(\infty) = p_i^*.$$

- As t falls to zero, we move from the discrimination case to the non-discrimination case. As t falls, λ rises. (In fact, if you compare the FOCs, when λ reaches 1, the FOC for the discrimination case coincides with the FOC for the artificial problem).

- We now want to show that when the constraint binds ($= 0$), it is actually that prices are converging and we just don't have a lot of positive and negative terms canceling each other out. To do this, we can show that, in fact, all the terms in the constraint are non-negative. Consider the constraint,

$$\sum_{i=1}^{N} \pi_i'(p^*)(p_i - p^*) \leq t.$$

With reference to the graph in the notes [**G-1.4**], for both strong and weak markets, it is clear that this expression is positive.

$$\text{For Strong markets: } \underbrace{\pi_i'(p^*)}_{Positive} \underbrace{(p_i - p^*)}_{Positive} \geq 0.$$

$$\text{For Weak markets: } \underbrace{\pi_i'(p^*)}_{Negative} \underbrace{(p_i - p^*)}_{Negative} \geq 0.$$

It follows then that as $t \to 0$, all terms in the summation $\to 0$. And since the first term is not equal to zero, then the second term must go to zero. This means that $p_i \to p^*$.

- Next week we will examine the way output changes as t falls from ∞ to 0. The idea: The welfare impact will involve 2 terms:
  - 1) The first of which corresponds to transferring units of output from weak to strong markets. This term is unambiguous in sign. Banning discrimination raises welfare.
  - 2) The second term relates to the fact that total output will change. This is not necessarily welfare improving. We will soon characterize this term.

2 Week 2: 21 Jan - 25 Jan

2.1 Summing up the Price Discrimination Model from last week

- The FOC for the artificial problem:

$$\pi_i'(p_i) = \lambda \pi_i'(p^*) \; \forall \; i.$$

When $\lambda = 0$, this collapses to the FOC for the discrimination case:

$$\pi_i'(p^*) = 0.$$

When $\lambda = 1$, it coincides with the FOC for the non-discrimination case:

$$\pi_i'(p_i) = \pi_i'(p^*).$$

And since π_i is concave, π' is monotonic and so this implies that $p_i = p^*$, ie, the solution for the non-discrimination case.

- We would now like to analyze the effects of a change in t, our control variable, first on output and the on welfare. We're trying to get at the effects of discrimination in terms of welfare.

- Take the FOC for the non-discrimination case and note that this implicitly defines p^*:

$$\sum_{i=1}^{N} \pi_i'(p^*) = \sum_{i=1}^{N} [(p^* - c)q_i'(p^*) + q_i(p^*)] = 0.$$

Refer back to last week's notes to see where this comes from.

- Combine this with the FOC for the artificial problem:

$$\pi_i'(p_i) = \lambda \pi_i'(p^*) \; \forall \; i.$$

Thus we get,

$$\sum_{i=1}^{N} \pi_i'(p_i(t)) = 0.$$

- Now, take this final equation and differentiate with respect to t noting that π_i is a function of p_i and p_i is a function of t. Thus (*),

$$\frac{d}{dt}\left[\sum_{i=1}^{N} \pi_i'(p_i(t)) = 0\right] \equiv \sum_{i=1}^{N} [\pi_i''] \cdot p_i'(t) = 0.$$

- To write this explicitly, take the equation for $\pi(p_i)$ and find the first and second derivatives with respect to p_i:

$$\pi_i(p_i) = (p_i - c)q_i(p_i).$$
$$\pi_i'(p_i) = (p_i - c)q_i'(p_i) + q_i(p_i).$$
$$\pi_i''(p_i) = (p_i - c)q_i''(p_i) + q_i'(p_i) + q_i'(p_i) = (p_i - c)q_i''(p_i) + 2q_i'(p_i).$$

- Thus substituting this into the equation above (*):

$$\sum_{i=1}^{N}[(p_i - c)q_i''(p_i) + 2q_i'(p_i)] \cdot p_i'(t) = 0.$$

  Call this equation, (**).

- Now we will use this equation to compute the OUTPUT effect:

  - Define total output as $Q = \sum_{i=1}^{N} q_i(p_i)$ where $p_i = p_i(t)$. Therefore,

$$\frac{dQ}{dt} = \sum_{i=1}^{N} \frac{dq_i}{dp_i}\frac{dp_i}{dt} = \sum_{i=1}^{N} q_i'p_i'.$$

    And substituting this from the equation (**),

$$\sum_{i=1}^{N}[(p_i - c)q_i''(p_i) + 2q_i'(p_i)] \cdot p_i'(t) = 0.$$
$$\sum_{i=1}^{N}(p_i - c)q_i''p_i' + 2q_i'p_i' = 0.$$
$$\sum_{i=1}^{N}(p_i - c)q_i''p_i' + 2\sum_{i=1}^{N} q_i'p_i' = 0.$$
$$2\sum_{i=1}^{N} q_i'p_i' = -\sum_{i=1}^{N}(p_i - c)q_i''p_i'.$$
$$\sum_{i=1}^{N} q_i'p_i' = -\frac{1}{2}\sum_{i=1}^{N}(p_i - c)q_i''p_i'.$$

    Thus,

$$\frac{dQ}{dt} = -\frac{1}{2}\sum_{i=1}^{N}(p_i - c)q_i''p_i'.$$

  - Note that in this expression, we have $q_i''(p_i)$ which is the second derivative of the market demand schedule. Thus if the demand schedule is linear, then the output effect is zero.

- Now consider the WELFARE effect:

  - We measure welfare with the following expression, (from last week) :

$$\underbrace{W = \sum_{i=1}^{N} \left\{ \overbrace{\int_{p_i}^{\infty} q_i(p)dp}^{CS} + \overbrace{\pi(p_i)}^{Profit} \right\}}_{Equation\ 4}.$$

  - Now to determine how welfare changes with changes in t, take the differential:

$$\frac{d}{dt}W = \frac{d}{dp_i}\{\cdot\}\frac{dp_i}{dt}.$$

  - Note that $\frac{d}{dp_i}\int_{p_i}^{\infty} q_i(p)dp = -q_i(p_i)$. Also note that $\pi_i' = (p_i - c)q_i'(p_i) + q_i(p_i)$.

$$\frac{d}{dt}W = \sum_{i=1}^{N}[-q_i + (p_i - c)q_i' + q_i]p_i' = \sum_{i=1}^{N}(p_i - c)q_i'p_i'.$$

  - To better understand the effects on welfare, break the equation into two parts,

$$\frac{d}{dt}W = \sum_{i=1}^{N}(p^* - c)q_i'p_i' + \sum_{i=1}^{N}(p_i - p^*)q_i'p_i'.$$

$$\frac{d}{dt}W = (p^* - c)\sum_{i=1}^{N} q_i'p_i' + \sum_{i=1}^{N}(p_i - p^*)q_i'p_i'.$$

$$\frac{d}{dt}W = \underbrace{(p^* - c)\frac{dQ}{dt}}_{Postive} + \sum_{i=1}^{N}(p_i - p^*)\underbrace{q_i'}_{Negative}p_i'.$$

  We have shown previously that the other two terms depend on if the markets are strong or weak:
  If Strong:

$$\frac{d}{dt}W = \underbrace{(p^* - c)\frac{dQ}{dt}}_{+} + \overbrace{\underbrace{\sum_{i=1}^{N}(p_i - p^*)}_{+}\underbrace{q_i'}_{-}\underbrace{p_i'}_{+}}^{-}.$$

If Weak:

$$\frac{d}{dt}W = \underbrace{(p^* - c)\frac{dQ}{dt}}_{+} + \overbrace{\underbrace{\sum_{i=1}^{N}(p_i - p^*)}_{-}\underbrace{q_i'}_{-}\underbrace{p_i'}_{-}}^{-}.$$

- Summing up: reducing t to zero is equivalent to banning discrimination. The second term represents the fact that (if the output effect is zero), then welfare is raised by banning discrimination.
- Refer to the illustration in the notes [**G-2.1**] that shows the welfare loss in strong markets and the welfare gain in weak markets, from allowing discrimination. The change in welfare can be decomposed into the output effect and the deadweight loss.

2.2 Game Theory Background

- Game Theory: 3 objects. 1) Players 2) Strategies for each player 3) Payoff function which maps the set of strategies of all agents into agents i's payoff.
- Nash Equilibrium (NE): A set of strategies, one for each player, such that, given the strategies of its rivals, each player is using a strategy that maximizes his payoff (Optimal Reply or OR)
- In an example of firm price competition: The price the firm charges is NOT the strategy. The price the firm sets is the firm's action. A strategy, in this case, would be more of a "pricing plan" or a system of relative prices.
- The essence of non-cooperative game theory is that there are no binding agreements.
- Some games have no pure strategy NE. Hence the necessity for mixed strategies where players have probabilities of playing individual strategies.
- There may also be several optimal replies for a player in pure or mixed strategies.
- We call coordination games, games in which two NE might have the same payoffs and it is just a matter of making sure all players choose the same strategy.
- Then there are prisoner's dilemma type games which the NE is not pareto efficient.

2.3 Two Classic Examples

2.3.1 Cournot Equilibrium

- Start with a monopoly and add frims to model competition.
- Let N firms sell a homogeneous product, produced at zero cost.

- All firms face a simple market demand schedule:

$$p = a - bQ = a - b\sum_{j=1}^{N} q_j.$$

- The profit of firm i:

$$\pi_i = (a - b\sum_{j=1}^{N} q_j)q_i.$$

Note that prices are determined by market quantity, here indexed by j, and profits are then: this price multiplied by the individual firm's quantity, here indexed by i.

- Let each firm chooses output, q_i, taking rivals output levels as given. We seek a NE in quantities, q_i. Thus maximize profits with respect to q_i noting that q_i is one of the elements in the summation of the q_j's.

$$\frac{\partial \pi_i}{\partial q_i} = a - b\sum_{j=1}^{N} q_j - bq_i.$$

- Let us examine the case of a symmetric NE in which all firms choose the same output at equilibrium. Denote total output as $Q = Nq = \sum_{j=1}^{N} q_j$. So we have:

$$a - b\sum_{j=1}^{N} q_j - bq_i = a - bNq - bq = a - bq(N+1) = 0.$$

$$a - bq(N+1) = 0 \Rightarrow q = \frac{a}{b(N+1)}.$$

And the demand schedule:

$$p = a - b\sum_{j=1}^{N} q_j = a - bNq.$$

Substituting in for q,

$$p = a - bN[\frac{a}{b(N+1)}].$$

$$p = a - [\frac{aN}{(N+1)}].$$

$$p = \frac{a(N+1) - aN}{(N+1)} = \frac{a}{(N+1)}.$$

- So in the case where this is only 1 firm ($N = 1$), the monopoly case, $p = \frac{1}{2}$, or the familiar $MR = MC$ result that gives us a price half way up the demand schedule.
- As $n \longrightarrow \infty$, $p \longrightarrow 0 = MC$ in this case.
- As more and more firms enter the market, prices fall to MC. When there are many firms in a market, each firm has a small share of the market so when the price falls, the quantity increase in demand per firm is large. Thus the incentive for prices to fall when there are many firms in an industry. Bertrand had other ideas.

2.3.2 Bertrand Equilibrium

- Consider an industry with N firms producing a homogeneous product. We seek a NE in prices known as a Bertrand Equilibrium. The strategy for firm i is P_i.
- The payoff functions for 2 firms look as follows:

$$\begin{cases} \text{if } P_1 < P_2 & \pi_1 = P_1 Q(P_1) \\ \text{if } P_1 > P_2 & \pi_1 = 0 \\ \text{if } P_1 = P_2 & \pi_1 = \frac{1}{2} P_1 Q(P_1) \end{cases} \tag{1}$$

  [G-2.3]

- Note that the payoff function is discontinuous so we use a graphical representation to look for NE. The only NE is when $P_1 = P_2 = 0 = MC$. Otherwise one firm or the other has an incentive to either undercut to steal the market, or if one firm is charging a zero price and the other is charging some positive price, then the zero price firm has the incentive to raise his price up to just below the price of his rival. The firm charging the positive price is indifferent because he is making zero profit either way. Once the other guy's price is above zero however, he then has an incentive to undercut which takes us back down to the stable equilibrium at $(0, 0)$.
- The Bertrand example is really not a good example of actual firm behavior because it does not take into account reactions. If one firm undercuts on one day and the other will undercut the following day, then this knowledge might cause the first firm NOT to undercut in the first place. This however, changes the dynamics of the game.
- Final remark: So far, all our examples are ones in which each firm chooses some single strategy from its strategy set. Consider a game that has no equilibrium in what we call "Pure Strategies." ie, a firm cannot just choose one pure strategy all the time and expect it to always be his optimal reply. Thus the need to introduce mixed strategies. Players play strategies with probabilities: the set of strategies is called the support.

3 Week 3: 28 Jan - 1 Feb

3.1 Mixed Strategy Equilibrium

- Consider the setup with the ship and sub deciding to go north or south around an island with payoffs of $(+1, -1)$ or $(-1, +1)$ depending on if they meet or not.
- Let p equal the probability that the sub goes north. If the ship goes north, its expected payoff is:

$$E[\pi] = p(-1) + (1-p)(+1) = 1 - 2p.$$

- If the ships goes south, its expected payoff is:

$$E[\pi] = p(+1) + (1-p)(-1) = 2p - 1.$$

- If both appear in the support of the stragegy, then,

$$1 - 2p = 2p - 1 \Longrightarrow p = \frac{1}{2}.$$

- We could do a similar calculation for the sub; if the ship goes north with probability $q = \frac{1}{2}$, then the sub is indifferent between going north and south. Thus we have a NE in mixed strategies where $p = q = \frac{1}{2}$.

3.2 Examples of Nash Equilibrium

3.2.1 Auctions

- The design of auctions is crucial and the application of game theory to auctions has been very successful. Consider the auctioning off of Oil tracts in the sea.
- A tract is defined as some geographical area in the water where there might be an adjacent tract that is already producing oil. Since it is very possible that the oil reserves below the sea are connected, the current owner of the adjacent oil tract will have superior information about the tract that is up for auction. Suppose there is one insider in the adjacent tract that knows exactly the value of the tract that is up for sale. The other bidders, the outsiders, do not know the true value of the tract.
- If the outsider bids too high, above the true value, then he might get an unprofitable tract. However, though it might seem that just not bidding is an optimal strategy for the outsider, it turns out that it is not. Consider if only the insider bids on the new tract. He will of course bid very low knowing that he will automatically win. But the outsiders would have a dominate strategy of bidding just above the insider and winning the tract. Thus, in a NE, the outsiders MUST bid.

- Consider an example of 2 bidders: an informed bidder and an uninformed bidder. The true value of tract to the bidder is denoted v where v is drawn from a uniform distribution on $[0, 1]$. Since both players would get the same value out of the tract, this type of auction is called a "common value auction."

- Assume that the informed bidder knows the exact value of v. The uninformed bidder only knows that $v \sim U[0, 1]$.

- We seek a Nash Equilibrium in bidding strategies.

- A strategy for the uninformed player is a bid, b. A mixed strategy specifices a probability distribution from which b is drawn.

- A strategy for the informed player takes the form of a function which maps the true value, v, into a bid $s(v)$. Payoff is the true value minus the bid, $v-$"bid."

- Since the derivation is rather complicated, we will now state the result and show it is Nash. The following strategies form a NE in mixed strategies:

  - The informed player bids $s(v) = \frac{1}{2}v$.
  - The uninformed player bids b, where b is drawn from a uniform distribution on $\left[0, \frac{1}{2}\right]$.

- Proof that this combination of strategies is Nash:

  - Uninformed player's strategy. Here we show any bid in $\left[0, \frac{1}{2}\right]$ yields expected payoff of zero.
  - To show that the uninformed strategy is Nash, assume that the informed player bids $\frac{1}{2}v$. Therefore, whatever the uninformed player bids, he will win if:

$$b > \frac{1}{2}v.$$

    Or,

$$v < 2b.$$

  - So consider the graph in the notes of the distribution of v on the uniform distribution $[0, 1]$. If $v > 2b$, then the informed player wins the auction and the payoff to the uninformed player is zero. If $v < 2b$, the uninformed player wins. So for $0 < v < b$, the payoff $v - b < 0$. For $b < v < 2b$, $v - b > 0$. Noting that these two areas are equal because the distribution is uniform [**G-3.1**], then the expect payoff is 0. So overall, the expected payoff for the uninformed player is zero.

  - Informed player's strategy. Given the strategy of the uniformed player, (bidding $b \in U\left[0, \frac{1}{2}\right]$), the expected payoff from an informed player bid of s is:

    * Win with probability 1 if $s > \frac{1}{2}$. Clearly b is never greater than $\frac{1}{2}$, so the informed player will always win.
    * Win with probability $2s$ if $0 \leq s \leq \frac{1}{2}$. This is because the uniformed player's bid is drawn from a uniform distibution on $\left[0, \frac{1}{2}\right]$. Given a bid, s, by the informed player in that range, the probability that $s > b$, is the probability that $b \in (0, s)$. Because the distribution is uniform this probability equals $2s$. **[G-3.2]**

  - Hence the expected payoff is:

$$E[\pi] = 2s(s - v) = 2s^2 - 2sv.$$

  - To determine the optimal bid, take the FOC:

$$\frac{d}{ds}E[\pi] = 4s - 2v.$$

  Setting equal to zero,

$$4s - 2v = 0 \Longrightarrow 4s = 2v \Longrightarrow s = \frac{1}{2}v.$$

  - We have confirmed that it is optimal, given the bidding strategy of the uninformed player, for the informed player to bid one half the true valuation.

- Two extensions of this model:
  - Many uninformed bidders: same outcome as above but the calculation focuses on the maximum bid from any uninformed bidder.
  - v is distributed on $[-x, 1]$, or some tracts are unprofitable. Here the informed player makes no bid if $v < 0$, so the uninformed player always "wins." (though of course, he still loses). However, the expected payoff to the uninformed players remain zero as before.

- Evidence: Hendricks and Porter Article. They studied the data to test predictions. Their predictions were as follows:
  - $\bar{\pi}$ of the informed player greater than zero.
  - $\bar{\pi}$ of the uninformed player equal to zero, as we showed above in the NE.
  - Profitability of the tracts, considering that an informed player will never place a bid if $v < 0$:

$$E[\pi = v - b | \text{ Uninformed Player Wins and Informed Player Does Not Bid}] < 0.$$

$$E[\pi = v - b | \text{ Uninformed Player Wins and Informed Player Does Bid}] > 0.$$

$$E[\pi = v - b | \text{ Uninformed Player Wins}] = 0.$$

- After running regressions on the data to determine if their predictions held up they found:

	Informed Player Wins	Uninformed Player Wins
$\pi = v - b$	6.76	-0.42
Standard Error	3.02	1.76
	Uninformed Wins and Informed does not Bid	
$\pi = v - b$	-2.69	
Standard Error	0.86	
	Uninformed Wins and Informed does Bid	
$\pi = v - b$	0.78	
Standard Error	2.64	

3.2.2 A Second Application: Repeated Games of Price Competition

- The repeated game or supergame framework is as follows: Let G denote the initial constituent stage game.
- We distinguish between finite and infinite horizon games denoted G^T and G^∞ respectively.
- The payoff function from G^T is given as the sum of the payoffs in G. The payoff of G^∞ is given by the NPV of the discounted flow of payoffs:

$$\sum_{t=0}^{\infty} \delta^t \pi^t \quad \delta \in [0,1].$$

- A strategy in G^T or G^∞ is a rule that specifies an action or strategy in G at time t as a function of the history of play from time 0 to time $t-1$. ie, the actions taken by all players prior to time t.
- Note that one equilibrium would be to ignore history: In G^T or G^∞, an equilibrium can be constructed by taking any NE of the constituent game and having each player play the constituent game strategy in every period.
- Example: Let G be the simple Bertrand model and consider G^∞.
  - Recall that the only NE of G is the one were each firm sets $P = MC$ so $\pi = 0$. We know from above that having each firm playing this strategy in every period constitutes a NE of G^∞.
  - Result: We can support any positive level of profits (such as monopoly profits) for a sufficent discount factor.
  - We need to build a rule into the game of pricing strategies when one firm deviates.

- In the first period set $P = P^M$, the monopoly price. In subsequent periods,
  - If rival's price in all previous periods has been at least P^M, set $P_t = P^M$.
  - If at any period in the past, the rival's price has been below P^M, set $P_t = MC$.
- We aim to show that this is a NE. Consider a situation where we have n firms and each share the monopoly profit equally if all set the monopoly price.
- FOLK THEOREM: Along the equlibrium path of the game, the payoff (NPV of profit flows), is:

$$\Pi = \frac{1}{n}\pi^M + \delta\frac{1}{n}\pi^M + \delta^2\frac{1}{n}\pi^M + \dots.$$
$$\Pi = \frac{1}{n}\frac{1}{1-\delta}\pi^M.$$

  Now consider the profit stream from a deviant firm who (WLOG) deviates at time $t = 0$. The expected payoff for the deviant is:

$$\Pi' \leq \pi^M + 0 + 0 + \dots.$$

  Thus, to make sure that deviation does NOT occur, it must be that $\Pi > \Pi'$. Thus,

$$\frac{1}{n}\frac{1}{1-\delta}\pi^M > \pi^M.$$
$$\frac{1}{n}\frac{1}{1-\delta} > 1.$$
$$\frac{1}{1-\delta} > n.$$
$$\frac{1}{n} > 1-\delta.$$
$$1 - \frac{1}{n} < \delta.$$

  So as long as δ is greater than $1 - \frac{1}{n}$, we maintain cooperation. This is intuitive because to avoid deviation now, we have to make sure that the expected profits in the future carry sufficiently high weight to dissuade a deviant firm from undercutting.
- Note on nomenclature: A trigger strategy specifies two actions and a rule for switching between the two. A grim trigger like we have here allows for no second chances.
- Note also that as $n \longrightarrow \infty$, $\delta \longrightarrow 1$, so deviation is more profitable and δ must be very high to maintain cooperation.

3.3 Beyond Nash Equilibrium

- Consider the possible set of strategy combinations and we now have developed a subset of this set called Nash Equilibrium. However, as is often a problem with NE, there are usually many of them. So, we seek a subset of Nash Equilibrium by introducing new restrictions and narrowing the number of possible equilibria.

- Consider a simple prisoners dilemma with strategies L_1 and M_1 for player 1 and L_2 and M_2 for player 2. The pareto efficient outcome is $(10, 10)$ from both players cooperating and playing L. The NE is both players playing M and receiving $(3, 3)$. The dilemma is held up by the off diagonal entries of $(11, 0)$ and $(0, 11)$. See game in notes. [**G-3.3**]

- Consider the finitely repeated game, G^T, and notice the following argument: the only NE of G^T is where (M_1, M_2) is played in each period. [**G-3.4**]

- More on this next week ...

4 Week 4: 4 Feb - 8 Feb

4.1 Subgame Perfect Nash Equilibrium

- Consider the game, G, as shown in the notes. **[G-4.1]** It is an extended prisoner's dilemma and the only NE in G is (M_1, M_2).

- We proceed to consider the game G^2 in which G is played in two successive periods. The payoff for G^2 is the sum of the payoffs in G.

- Question: What are the outcomes that can be supported as a NE in the two-period game? Clearly, one NE is formed by playing the NE in G in both periods. But can we form another NE of G^2 where the virtuous outcome, playing (L_1, L_2), is achieved in the first period? The answer is YES. A pair of strategies that support this is as follows: Player i plays L_i in period 1, and then follows the rule in period 2:

$$\text{Play in period 2:} \begin{cases} M_i \text{ if } (L_1, L_2) \text{ was played in period 1} \\ R_i \text{ otherwise} \end{cases} \tag{2}$$

- Playing the R strategy after the deviation is sort of a threat point that prevents anyone from deviating in period 1.

- Proof: Note that along the equilibrium path of the game, the outcome is (L_1, L_2) in period 1 and (M_1, M_2) in period 2 and each player gets payoff $= 10 + 3 = 13$. By deviating, (to M_i), in period 1, the payoff to the deviant is $11 + 0 = 11$. Thus deviation is not profitable. QED.

- However, the NE we just created is NOT perfect.

- Define: (Subgame) Perfect Nash Equilibrium (SPNE). A set of strategies that form a NE, and which induce a NE in every subgame of the game. In a game of complete information the term "Subgame PNE" is equilivalent to "PNE."

- In the game G^2, we can find the SPNE by means of a backward induction. ie, first consider the game G played at period 2. The only NE is (M_1, M_2), with payoffs equal to $(3, 3)$. Now analyze period 1. The payoff as of period 1, are those payoffs of G plus the payoffs in period 2, $(3, 3)$. Thus, the only equilibrium in G^2 at period 1 is to play (M_1, M_2). So (M_1, M_2) in period 1 and (M_1, M_2) in period 2 is the only SPNE.

4.1.1 An Example of SPNE

- Consider the extensive form of the game shown in the notes. We define graphically nodes, subgames, terminal nodes, and the payoffs to the players.

- Now consider the extensive form game where player 1 plays first and plays L_1 or R_1. **[G-4.2]** If he plays L_1, the payoffs to players 1 and 2 are $(0, 2)$ respectively. If player 1 plays R_1, then player 2 gets to play either l_2 or r_2 with payoffs $(-1, -1)$ or $(1, 1)$ respectively.

- Note the definition of NE: A set of strategies such that, given the strategies of the rival(s), each player is using an optimal reply (ie, there is no alternative strategy that yields a strictly higher payoff).

- In the game just described, there are two NE: (R_1, r_2) and (L_1, l_2). It is clear that the first NE is also SPNE which is easily shown using backward induction. But why is (L_1, l_2) a NE? Given that 1 is playing L_1, 2 gets a payoff of 2 whether he plays l_2 or r_2. So player 2 is using *an* optimal reply. (There is no alternative strategy that yields a strictly higher payoff). Given that 2 is playing l_2, player 1 gets 0 if he plays L_1 and -1 if he plays R_1. So playing L_1 is optimal. Thus (L_1, l_2) is a NE but not perfect.

- Intuition: imposing the additional restriction of perfectness excludes empty threats (or promises). Player 2 saying that he will play l_2 if 1 plays R_1 is an empty threat because once 1 plays R_1, player 2 will not want to play l_2, because it is in his interest to play r_2.

- SPNE is a good restriction that we can apply to limit the number of NE. Other restrictions that have been brought forward have not received as much positive support.

4.1.2 Technical footnotes

- Consider two games: In game G, player 2 knows which node he is at because he observes player 1's action. In game G', the players make simultaneous moves. See notes for extensive and normal forms.

- In game G, player 2 does not have only 2 strategies, but rather 4. His strategies take the form of a function (mapping) of 1's action into an action for player 2. We can write 2's strategies in the form (XY) where X is 2's reply to L_1 and Y is 2's reply to R_1. So overall player 2 has the following 4 strategies: (l_2, l_2),(l_2, r_2),(r_2, l_2), and (r_2, r_2).

- To write the game G' in extensive form, we must introduce the concept of information sets. **[G-4.3]** An information set is a set of nodes such that the player taking an action at these nodes, must take the same action at each node in the set. The intuition is that the player cannot observe which node he is at. See the graph in the notes that shows how we graphically represent an information set. The key idea is that player 2 must play either l_2 at both nodes or r_2 at both nodes. Player 2 cannot condition his strategy.

- A subgame MUST NOT break information sets.

4.2 Non-cooperative Bargaining Theory

- Two players aim to divide a cake of size 1. We will consider simultaneous games first.

- Simultaneous Move Games.

- Player 1 proposes to take a share of the cake equal to x. Player 2 proposes to take a share of the cake equal to y. The payoffs are as follows:

$$\text{Payoff:} \begin{cases} (x,y) \text{ if } x+y \leq 1 \\ (0,0) \text{ if } x+y > 1 \end{cases} \tag{3}$$

- Result: An partition (x,y) where $x+y=1$ can be supported as a NE. Proof: Given a demand of x by player 1, player 2 receives y so long as $y \leq 1-x$. Hence $y = 1-x$ is the optimal reply and similarly for player 1.
- This results captures the so-called "Indeterminacy of the bilateral monopoly."

• A Sequential One-Period Game.

- Let player 1 make a proposal (x,y) with $x+y \leq 1$ and player 2 replies yes or no.
- The payoffs are as follows:

$$\text{Payoff:} \begin{cases} (x,y) \text{ if yes} \\ (0,0) \text{ if no} \end{cases} \tag{4}$$

- This is called the "Ultimatum Game."
- Result 1: Any division (x,y) can be supported as a NE of this game. To see this, note that 2's strategy in this game takes the form of a function (mapping) from a proposal by 1 into a yes/no response by player 2. To support (x,y) as a NE, use appropriate strategies as follows: player 1 proposes (x,y) and the strategy of player 2 is as follows:

$$\text{Reply:} \begin{cases} \text{Accept if 2 gets at least } y \\ \text{Reject if 2 gets less than } y \end{cases} \tag{5}$$

However, this clearly IS NOT subgame perfect.
- Result 2: There is a unique perfect NE in this game. To see this analyze the subgame beginning with 2's reply. A NE in this subgame requires 2 to make a response that maximizes 2's payoff. If 1 offers (x,y) with $y > 0$, then the only NE in this subgame is one where 2 says yes. If 1 proposes $(1,0)$, then 2 is indifferent, so both yes and no constitute an optimal reply.
But this implies that any (x,y) with $y > 0$ cannot be supported as a PNE because 1 can always do better by offering $\frac{1}{2}y$ (and earn a strictly higher payoff).
So we have only one candidate equilibrium, $(x,y) = (1,0)$. This can be supported as a PNE using the following strategies: Player 1 proposes $(1,0)$ and player 2 always says "yes." Note that 2 is indifferent between "yes" and "no" given 1's offer.

• Sequential Multi-period game.

– The structure of the game is displayed in the notes where at time $t = 0$, player 1 proposes (x, y). **[G-4.4]** If player 2 accepts, the payoffs are (x, y). If player 2 says no, then in period $t = 1$, player 2 can propose a division (x, y). If player 1 accepts the division is (x, y), but the payoffs are now $(\delta x, \delta y)$. Then in period $t = 2$, if we get agreement, payoffs are $(\delta^2 x, \delta^2 y)$. This goes on and on so in general, agreement at time $= t$ yields payoffs $(\delta^t x, \delta^t y)$.

– Motivating idea: It might seem natural to analyze a finite horizon where the game ends at time T. (Failure to say yes by time T implies payoffs $(0, 0)$.) You can show there is a unique SPNE and that this coincides with the SPE of the infinite horizon game. In general, it is not true that the SPNE outcomes of an infinite horizon game coincides with the limit points of the SPNE outcomes of the finite games as $T \to \infty$. See picture in notes that shows that the limit points of the SPNE outcomes in a finite horizon are a subset of the SPE outcomes of the infinite horizon games. **[G-4.5]**

– The key aim in the bargaining problem: is there a unique equilibrium? To prove uniqueness in the infinite horizon game, we need to analyze it directly.

– A direct analysis can be constructed as follows: Note following a “No” in periods 0 and 1, we are left with a game that is identical in structure to the original game. (Because then in period 2, player 1 gets to make the offer as he would do in period 0). We will use this to show (next week) there is a unique perfect equilibrium outcome (via Rubenstein.)

5 Week 5: 11 Feb - 15 Feb

5.1 Non-Cooperative Bargaining Theory

- We aim to examine an infinite horizon sequential bargaining game.
- The game: At time 0, player I makes a proposal and player II replies. If player II says no, then the game will continue. At time 1, player II proposes and player I replies. If player I says no, then at time 2, player I again makes a proposal and player II replies. Etc.
- The discount factor of player i is δ_i, where $0 < \delta_i < 1$.
- To construct a perfect equilibrium, we usually start at the end and work backwards, but clearly in an infinite horizon game, we must proceed differently. The key will be that the game starting at time 0 has the same structure as the game starting at time 2.
- There may exist many equilibria, so we will look only at the extreme cases first. It will turn out that the extreme cases are the same so we will get a unique equilibrium.
- Let M denote the supremum of the share which player I can obtain in any perfect equilibrium of the game. (Note supremum is a number that cannot be exceeded.)
- Consider the following table:

Time	Offer Made By	I gets at MOST	II gets at LEAST
0	I	$1 - \delta_2(1 - \delta_1 M)$	
1	II		$1 - \delta_1 M$
2	I	M	

The explanation of this table is as follows. Start in time 2, and consider that player I gets M at most by definition. In period 1, player II gets to make an offer. He knows he only has to offer player I at most M, but since I can get M tomorrow, effectively in this period, player II only needs to offer player I, $\delta_1 M$. This makes player I indifferent between accepting $\delta_1 M$ in period 1 or waiting at getting M in period 2 (which will only be worth $\delta_1 M$ by that time.) Thus, if player II offers $\delta_1 M$ to player I, then player II himself gets $1 - \delta_1 M$. Now consider time 0. Player I makes the offer and he knows that he needs to offer $1 - \delta_1 M$ to player II, but again we must discount this value by δ_2 (player II's discount value) because player I only needs to offer $\delta_2(1 - \delta_1 M)$ to player II because getting $1 - \delta_1 M$ in period 0 will only be worth $\delta_2(1 - \delta_1 M)$ once we get to period 1. If player II is offered $\delta_2(1 - \delta_1 M)$ in period 0, then player I gets $1 - \delta_2(1 - \delta_1 M)$ in period 0.

- Now, since we know the structure of the game in period 0 is identical to the game starting in period 2, the payoffs for player I must be the same. Thus,

$$1-\delta_2(1-\delta_1 M) = M.$$
$$M-\delta_1\delta_2 M = 1-\delta_2.$$
$$M(1-\delta_1\delta_2) = 1-\delta_2.$$
$$M = \frac{1-\delta_2}{1-\delta_1\delta_2}.$$

- We can now repeat this argument, beginning by defining M as the infimum of player I's share and relabling the table as "I gets at LEAST" and "II gets at MOST." This process of course yields exactly the same equation so the equation for M above is unique.

- Thus $M = \dfrac{1-\delta_2}{1-\delta_1\delta_2}$ defines a unique perfect equilibrium partition of the cake. (Rubinstein).

- Remarks on this solution. At equilibrium, player I gets $\dfrac{1-\delta_2}{1-\delta_1\delta_2}$. If we set $\delta_1=\delta_2=\delta$, then player I gets $\dfrac{1-\delta}{1-\delta^2} = \dfrac{1-\delta}{(1-\delta)(1+\delta)} = \dfrac{1}{1+\delta}$. Thus player II gets $1-\dfrac{1}{1+\delta} = \dfrac{\delta}{1+\delta}$. So we get an equilibrium allocation of:

$$\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right).$$

  Player I, who moves first, gets more than player II but as $\delta \to 1$, meaning that as the players get infinitely patient, then the first mover advantage disappears and the shares converge to $(\frac{1}{2},\frac{1}{2})$.

- Once we add more than 2 players, we lose uniqueness.

5.2 Part II: Product Differentiation

- Firms never face flat demand schedules, there is also some degree of heterogeneity in products.

- Hotelling model (1929). Transport costs makes otherwise homogeneous goods to be horizontally differentiated. See graph in notes. **[G-5.1]**

- Utility function of representative consumer: $U =$ Constant $-p - t \cdot d$, where p is the price of the good, t is the per unit transport costs (or the degree of preference intensity for slightly differentiated products), and d is the distance traveled.

- Step 1: Given the locations of the firms, we want to analyze the price competition and find the equilibrium profit of each firm. Step 2: The choice of locations by firms. We set this up with a 2 stage game where in stage 1, firms choose a location and in stage 2, firms compete in prices.

5.2.1 A Preliminary Example - The Circular Road Model

- This type of model gets around the problem of firms on the "ends." See graph in notes. **[G-5.2]**
- Assume there are N firms located uniformly around the circular road of circumference 1. Thus the distance between any two firms is $\frac{1}{N}$. Assume also that firms only travel around the edge of the circle.
- Let $MC = 0$ for all firms.
- We seek a NE in prices. In fact, we will seek a symmetric NE.
- Denote $\bar{p}$ as the equilibrium price. Consider a deviant firm who sets a price, p.
- See graph in notes which shows Hotelling's Umbrellas, but we have a situation where the deviant's umbrella looks slightly different than the neighboring umbrellas. **[G-5.3]** Define the "Marginal Man" as the person who is located between the deviant firm and one of the neighboring firms. Note this does not mean that he is right in the middle, but rather falls just under the intersection of the two umbrellas. Define the distance to the deviant as d. Thus, the distance to the neighboring firm is $\frac{1}{N} - d$.
- So we have the equation of the marginal man. (His utility gained from going to either firms is the same).

  $$\underbrace{p + t \cdot d}_{Utility\ from\ Deviant\ Purchase} \quad = \quad \underbrace{\bar{p} + t(\frac{1}{N} - d)}_{Utility\ from\ Neighboring\ Purchase} \quad .$$

  Rearranging,

  $$p + td = \bar{p} + t(\frac{1}{N} - d).$$
  $$p - \bar{p} - \frac{t}{N} = -td - td.$$
  $$2td = \bar{p} - p + \frac{t}{N}.$$

- Define the density of consumers around the circular road as 1. Thus each firm's sales equals the length of the relavent line segment. So for the deviant firm, sales:

  $$Sales = x = 2d.$$

$$x = \frac{\bar{p} - p}{t} + \frac{1}{N}.$$

- Profits. Recall that costs are zero. Thus,

$$\pi = px = p(\frac{\bar{p} - p}{t} + \frac{1}{N}).$$

- Differentiating,

$$\frac{d\pi}{dp} = \frac{\bar{p} - p}{t} + \frac{1}{N} - \frac{p}{t}.$$

  Now, for a symmetric NE, we aim to find a value $\bar{p}$ such that when $p = \bar{p}$, the above derivative equals 0. So letting $p = \bar{p}$ and setting equal to zero,

$$\frac{\bar{p} - p}{t} + \frac{1}{N} - \frac{p}{t} = 0.$$

$$\frac{\bar{p} - \bar{p}}{t} + \frac{1}{N} - \frac{\bar{p}}{t} = 0.$$

$$\frac{1}{N} = \frac{\bar{p}}{t}.$$

$$\bar{p} = \frac{t}{N}.$$

- Interpretation: as $t \to 0$, $\bar{p} \to 0 = MC$. We converge to the homogenous product case and our model is actually the Bertrand model of price competition. For any fixed $t \neq 0$, note that as $N \to \infty$, $\bar{p} \to 0$. Even though transport costs are not zero, each firm must have closely similar neighbors.

- Finally we examine the demand schedule faced by the firm at equilibrium, ie, when rivals set $\bar{p} = \frac{t}{N}$. See graphs in notes that display that as p goes to zero, the deviant firms takes the entire market from its neighbors. [**G-5.4**]The edges of the umbrella intersect exactly with the neighboring umbrellas. As the price of the deviant rises to the point where it loses the whole market, this can also be shown via the umbrellas because the deviant firms "coverage" collapses to zero as its neighbor's umbrellas cover the customers in the interval.

- If we have a situation where as the deviant firm lowers his price and takes over his neighbor's entire customer interval before his price reachs zero, then all of a sudden, the deviant firm will have a discontinuous jump in product demand when he starts taking over customers on the "other side" of his neighbor. See graph in notes for this analysis. [**G-5.5**] The demand curve is therefore discontinuous in this case.

5.2.2 The Choice of Location

- Some motivating examples. Assume there are no price decisions. We have a 1 shot game.
- Each firm chooses a location on the line segment $(0, 1)$. A firm's payoff is given by the length of the line segment consisting of points closer to the firm than to the other firms.
- If 2 or more firms have the same location, they share equally the payoff associated with that location.
- We aim to find a NE in locations, for a given number of firms, N.
- Let $N = 2$. Player I chooses x and player II chooses y. To find a NE, first consider the case $x \neq y$. This is clearly not Nash because player I can gain a higher payoff by moving towards y. So the NE involves $x = y$. Suppose $x = y \neq \frac{1}{2}$, then this is also not Nash. Either firm can raise its payoff by moving slightly towards the long end of the line. Thus the only NE is $(x, y) = (\frac{1}{2}, \frac{1}{2})$. At this point, neither firm as a profitable deviation.
- Now let $N = 3$. Here there is NO NE in pure strategies! Consider 3 firms organized from left to right, x, y, and z. Consider the cases: If $x \neq y \neq z$. Not Nash because x should move towards y. Suppose $x \neq y = z$ or $x = y \neq z$. Again this is not Nash because x can raise payoff by moving right towards y and z. Let $x = y = z$. Here all profits are equal to $\frac{1}{3}$. Any firm can achieve a higher payoff by moving slightly towards the Long end of the line (or if they're in the middle, then moving in either direction is profitable.)
- For $N = 4$ or above. The equilibrium configuration is of the form where two pairs firms are positioned together somewhere on the line and the other firms are distributed equally among the remaining line segment. For $N = 4$, the two pairs of firms are positioned at $\frac{1}{4}$ and $\frac{3}{4}$ along the line. It is fairly easy to show that no deviation from this configuration is profitable.
- For $N = 5$, two firms position themselves at $\frac{1}{6}$, one firm is at $\frac{1}{2}$, and the final two firms are positioned at $\frac{5}{6}$ along the line. Careful inspection verifies that these locations form a NE.

6 Week 6: 18 Feb - 22 Feb

6.1 The Modified Hotelling Model

- In the original 1929 Hotelling model, he modeled a two stage game where in stage 1, firms choose a location and in stage 2, we seek a NE in prices.

- Using a linear transport cost, there is a small problem which Hotelling dismissed in 1929 as a technicality. The problem is with the linear umbrellas. If lowering one firm's price makes his umbrella intersect exactly with the opposite branch of his neighbor's umbrella, the demand for the product become discontinuous. See notes from week 5 for picture of this. This problem occurs when firms are located closely together. Hotelling concluded that in the two firm model along a circular road, firms choose to both position themselves exactly at the center. This became known as the "Principal of Minimal Differentiation." However, consider this case if we have Bertrand price competition. In this case, prices are equal to marginal cost and profits are zero. Therefore if either firm moves a little towards the "ends," his product becomes differientiated so he makes positive profit. Thus, with this type of (severe) price competition Hotelling's NE is clearly NOT Nash.

- In 1979, 50 years after Hotelling's paper, d'Aspremont, Gabszewicz, et. al, took up this error and introduced a slightly different set up to show how we can get around this error. The results of their analysis are radically different than Hotelling's.

- We move now to a Quadratic Cost Function. (We could use a linear cost function and look for a mixed strategy equilibrium, but the analysis gets very complicated). The technical idea which will result is that this type of cost removes the discontinuity in the demand function that is present with linear cost.

- We set up the model as follows: firms A and B choose locations along a line of length, l. A chooses location a distance, a, from the left end of the line. B chooses location a distance, b, from the right end of the line. Thus, the distance between the two firms is $l - a - b$. See pictures in notes. **[G-6.1]**

- We write down a demand and profit schedule for each firm:

$$\pi_i(p_1, p_2; a, b) \; for \; i = 1, 2.$$

  We refer to his profit function as the payoff function for the stage 2 subgame where the locations, (a, b), are taken as fixed. (thus they are written following the ;).

- Our recipe is as follows: Solve for p_1 and p_2. Then insert these prices into the profit function and find the payoff functions for the stage 1 subgame as follows:

$$\pi_i(a, b) \; for \; i = 1, 2.$$

- Now consider the derivation of the demand schedule for the stage 2 subgame. See graph in notes. [**G-6.2**] We denote the cost function as follows:

$$p_i + cx^2.$$

This results in parabolic umbrellas for each firm. Note that because the slope of the umbrellas increases as you get farther away from the firm, you never get the case where one firm's umbrellas coincides exactly with a neighboring umbrella. There must exist exactly ONE intersection. This gets around the discontinuity. See graph in notes. [**G-6.3**]

- Define the following in terms of the Marginal Man (MM):

$$d_1 \equiv \text{ The signed distance of the MM to the right of } A.$$
$$d_2 \equiv \text{ The signed distance of the MM to the left of } B.$$

Thus, if the MM lies between firm A and B, then both d_1 and d_2 are positive. If the intersection, say, occurs to the left of firm A, (so firm A has no market share), then d_2 is positive, but d_1 is negative.

- The equation of the marginal man is therefore:

$$p_1 + cd_1^2 = p_2 + cd_2^2.$$

Therefore the utility from consuming from A is exactly equal the utility from consuming from B. Note the utility now involves a squared term of the distance from MM to each firm. Solving:

$$p_2 - p_1 = cd_1^2 - cd_2^2 = c(d_1^2 - d_2^2) = c(d_1 - d_2)(d_1 + d_2).$$

Note again that l is the length of the line so $a + b + d_1 + d_2 = l$, or $d_1 + d_2 = l - a - b$. Substituting,

$$p_2 - p_1 = c(d_1 - d_2)(l - a - b).$$

[Note that the fact that d_i is a signed distance, we can always make this substitution no matter where MM lies.] Solving for $d_1 - d_2$,

$$d_1 - d_2 = \frac{p_2 - p_1}{c(l - a - b)}.$$

Consider again the equality,

$$d_1 + d_2 = l - a - b.$$

Adding together the previous two equations,

$$d_1 - d_2 + (d_1 + d_2) = \frac{p_2 - p_1}{c(l - a - b)} + l - a - b.$$

$$2d_1 = \frac{p_2 - p_1}{c(l - a - b)} + l - a - b.$$

Thus,

$$d_1 = \frac{p_2 - p_1}{2c(l - a - b)} + \frac{l - a - b}{2}.$$

- Now the sales of firm A is equal to the customers to the left of A, a, plus the customers to right of A, d_1. **[G-6.4]** So,

$$q_1^* = a + d_1 = a + \frac{p_2 - p_1}{2c(l - a - b)} + \frac{l - a - b}{2}.$$

This holds iff $q_1 \in (0, l]$. If the above expression yields $q_1 < 0$, then $q_1^* = 0$. If the above expression yields $q_1 > l$, then $q_1^* = l$.

- From this demand schedule, we compute profits:

$$\pi_i(p_1, p_2; a, b) = p_i q_i(p_1, p_2).$$

And these are the payoffs in the stage 2 subgame. We then solve for a NE in prices by setting:

$$\frac{d}{dp_i}\pi_i(p_1, p_2; a, b) = 0.$$

Call this solution (p_1^*, p_2^*). Substituting these prices back into the profit function, we obtain the payoff function for the stage 1 subgame:

$$\pi_i(a, b).$$

- The final step is using these profit functions as payoff functions for the location choice game (ie the stage 1 game), we seek a NE in locations. The KEY RESULTS:

$$\frac{\partial \pi_1}{\partial a} < 0.$$

$$\frac{\partial \pi_2}{\partial b} < 0.$$

So what this says is if we decrease a for example, profits will rise. This involves moving towards the end of the line. Thus the only NE is $a = b = 0$ or where firms are located at opposite ends of the line. For lack of a better expression, this might be referred to as the "Principal of Maximal Differentiation." Note it is exactly opposite from the Hotelling solution.

- Beyond the d'Aspremont et. al. example. It seems that we have two competing ideas:
  - 1.) Moving towards the other firm in the market shifts over the MM and raises the market share and sales volume of the firm.
  - 2.) Moving away from the other firm makes the two goods less (horizontally) substitutable and increases prices and profits.
- In general, the NE in locations will reflect the interplay of these two ideas. The formulation of the cost function and the degree of price competition are important in determining where the NE will be. For example, less intense price competition (such as Cournot instead of Bertrand), leads to CLOSER NE locations.

6.1.1 Entry Decisions

- So far we have taken the number of firms in an industry, n, as given. A further issue is analyzing entry.
- Let firms incurr a setup cost for entry of $\epsilon > 0$. Let the density of consumers on the line enter as a parameter S, standing for the "Size of the market."
- The parameter, S, enters multiplicatively in the profit function. A necessary condition for entering is that:

$$S\pi(\cdot) \geq \epsilon.$$

- A general feature of an equilibrium with entry: outcomes depend on the ratio $\frac{S}{\epsilon}$. As $\frac{S}{\epsilon}$ rises, the density of firms on the line increases. (Either as the market size gets larger or the entry cost falls). [Note, all this assume single product firms.] Also, as $\frac{S}{\epsilon} \to \infty$, then C_1 (the concentration of the largest firm in the industry) $\to 0$.

6.2 Vertical Product Differentiation

- Consider two firms, A and B, who sell at the same price, $P_A = P_B = P$. In horizontal product differentiation (HPD), some consumers prefer A and some prefer B, but in vertical price differentiation (VPD), all consumers choose the same product, namely the higher quality item.
- Note that it is consumer's "perceived" quality that matters. Thus it is consumer's willingness to pay that will determine A and B's market shares.
- The model goes as follows: Assume a number of firms offer distinct, substitute goods which vary in quality. (Note that if the goods were homogenous, this would just collapse to a Bertrand game.)

- Consumers buy one unit or none.
- Firms have zero cost. (VERY important assumption)
- Goods labeled $k = 1...n$ and thus firm k sells product k at price p_k.
- Consumer's income levels are distributed uniformly over the interval $t = [a, b]$.
- Consumer's Utility level:

$$u(t, k) = u_k(t - p_k)$$
$$u(t, 0) = u_0 t.$$

Where t is the consumer's income level and,

$$0 < u_0 \text{ constant } < u_1 < \cdots < u_k < \cdots < u_n.$$

[G-6.5]

- Obviously, the consumer's income level is what determines his willingness to pay and thus the demand for each good.
- $u(t, 0)$ is what the consumer will get if he does not buy any of the n goods and spends his income and some other bundle of goods. This bundle is referred to as the "Hicksian Composite Commodity" and is defined by holding all relative prices of goods in other markets constant.
- Consider the "Marginal Man." He is just indifferent between buying good k and good $k-1$ which is a bit less expensive but also of lower quality. Consider graph in notes that shows that the price gap for the marginal man increases as his income rises. **[G-6.6]** Thus the more wealthy people are more willing to pay more for quality. Thus,

$$u_k(t_k - p_k) = u_{k-1}(t_k - p_{k-1}).$$

Where t_k is the marginal man's income level. He is just indifferent between paying p_k for u_k and paying p_{k-1} for u_{k-1}.

- Solving the marginal man's equation for t_k,

$$u_k t_k - u_{k-1} t_k = u_k p_k - u_{k-1} p_{k-1}.$$
$$t_k(u_k - u_{k-1}) = u_k p_k - u_{k-1} p_{k-1}.$$
$$t_k = \frac{u_k p_k - u_{k-1} p_{k-1}}{u_k - u_{k-1}}.$$

- Now define the C_k as:

$$C_k = \frac{u_k}{u_k - u_{k-1}}.$$

This quantity will show up in the algebra below so it will simplify things. Note that $C_k > 1$ because $u_{k-1} < u_k$. Also, as the quality of these two goods get very close, $C_k \to \infty$.

- Thus we can simplify the above equation to,

$$t_k = C_k p_k + (1 - C_k) p_{k-1}.$$

- FOCs:

$$\frac{\partial t_k}{\partial p_k} = C_k > 1.$$

$$\frac{\partial t_k}{\partial p_{k-1}} = 1 - C_k < 0.$$

- Now consider the highest quality good, good n. Let $t_n > a$ so that at least 2 goods survive in the market. (We avoid a corner solution in this case). Note that the top quality producing firm can always survive because he can always price the lowest and gain the whole market. Define the profits of the top quality firm:

$$\pi_n = R_n = p_n(b - t_n).$$

Because costs are zero, profits equal revenues. p_n is the price charged and $b - t_n$ is the fraction of the market that the top quality firm gets. Taking the derivative,

$$\frac{\partial \pi_n}{\partial p_n} = b - t_n - p_n \frac{\partial t_n}{\partial p_n}.$$

Substituting in from above,

$$\frac{\partial \pi_n}{\partial p_n} \Rightarrow b - t_n - p_n C_n = 0.$$

But also from above,

$$p_n C_n = t_n - p_{n-1}(1 - C_n).$$

Substituting in,

$$b - t_n - t_n + p_{n-1}(1 - C_n) = 0.$$

$$b - 2t_n + p_{n-1}(1 - C_n) = 0.$$

$$b - 2t_n - p_{n-1}(C_n - 1) = 0.$$

Since $C_n - 1 > 0$,

$$b - 2t_n > 0.$$

Or,

$$t_n < \frac{1}{2} b.$$

- So if $a > \frac{1}{2}b$, it cannot be that more than one firm survives. Here the lowest income consumer $(t = a)$, prefers to buy product u_n at price p_n rather than buy the next highest quality product, u_{n-1}, at price zero. Graphically, we get a corner solution where there is no room for an additional firm to enter. **[G-6.7]** An interior solution occurs more frequently when a is small so $b - a$ is large and it is likely that $a < \frac{1}{2}b$.

Thus the concentration of income becomes important because a more concentrated income base will only allow one quality of good in the market, namely the highest quality product. (See graphs). Note that the demand schedule for firm n depends on the price charged by firm $n-1$. As the price of firm $n-1$ rises, the demand for firm n's product rises. (as is shown in the graphs).

- Now consider the same argument for a general good, k. Profits for firm k,

$$\pi_k = R_k = p_k(t_{k+1} - t_k).$$

FOC:

$$\frac{\partial \pi_k}{\partial p_k} = t_{k+1} - t_k - p_k \frac{\partial t_k}{\partial p_k} + p_k \frac{\partial t_{k+1}}{\partial p_k}.$$

Substituting in from above,

$$\frac{\partial \pi_k}{\partial p_k} \Rightarrow t_{k+1} - t_k - p_k C_k + p_k(1 - C_{k+1}) = 0.$$

But also from above,

$$p_k C_k = t_k - p_{k-1}(1 - C_k).$$

Substituting in,

$$t_{k+1} - 2t_k - p_k(C_{k+1} - 1) - p_{k-1}(C_k - 1) = 0.$$

Since $C_{k+1} - 1 > 0$ and $C_k - 1 > 0$,

$$t_k < \frac{1}{2} t_{k+1}.$$

- It immediately follows therefore:

$$t_n < \frac{1}{2}b, \;\; t_{n-1} < \frac{1}{4}b, \;\; t_{n-2} < \frac{1}{8}b, \;\; t_{n-3} < \frac{1}{16}b, \;\; \ldots$$

- Thus, price competition between the high quality goods drive down their prices to a level such that at equilibrium prices, even the poorest consumer prefers to buy the last surviving good at its equilibrium price rather than the "best non-surviving" good at $p = MC = 0$.

- Result: The Finiteness Property: There exists a bound, B, INDEPENDENT OF QUALITY such that at most B firms can have positive sales revenues at a Nash equilibrium in prices. Thus,

$$\text{If } a > \frac{1}{2}b, \text{ then } B = 1.$$

$$\text{If } \frac{1}{2}b > a > \frac{1}{4}b, \text{ then } B = 2.$$

$$\text{If } \frac{1}{4}b > a > \frac{1}{8}b, \text{ then } B = 3.$$

$$\vdots$$

- In HPD, we had the situation where the more consumers that were in the market, the more room there was for another firm to enter the market. In VPD, There is no effect of more consumers in the market. This is an example of the non-convergence principal.

- In conclusion: Entry of high quality firms induce exit of low quality firms. The presense of firm's of similiar quality causes both p_i and p_j to fall to zero. So as the qualities of products in a market get close, the market share's of firms remain concentrated but prices go to zero.

7 Week 7: 25 Feb - 1 Mar

7.1 Some final Remarks on Vertical Product Differentiation

- The Key element is the mapping from unit variable (marginal) cost labelled c, to the quality level, u. ie, the mapping $c(u)$. In the example we did last week, $c(u) = 0$ for all u. Hence $c(u)$ would be perfectly horizontal.

- In our model we had the maintained hypothesis that the marginal cost of production (as function of output) was constant. Call this marginal cost c. The MC embodies things like labor and raw materials.

- Now consider the case where the mapping $c(u)$ is very steep. This means that it is very costly (per unit of quality) for a firm to increase quality. Consider an industry such that cost of increasing quality depends mostly on labor and materials, such as building a high quality table.

- There is also the case where $c(u)$ is flat which implies that the main burden on quality improvement falls on fixed costs like $R\&D$ and the like. The marginal costs do not rise as steeply in these types of industries.

- The steepness of this $c(u)$ line is the key to the analysis. See the diagram in the notes of the Hotelling setup (HPD) where all prices equal marginal cost. We get a simple 1:1 mapping between consumers and producers along the beach. The key, mathematically, is that we can define an open set of consumers on the line and map it into an open set on the producer line so that each consumer goes to exactly one unique producer.

- In the world of vertical product differentiation, qualities differ across products and assume the $price = c(u)$. If $c(u)$ is steep, the same 1:1 mapping can be formed because the open sets can be formed as in the hotelling case. This is because consumers have to pay a significant amount more for a quality improvement. So, mathematically, we can find an open interval of products, say, (u_1, u_2) and a corresponding open interval of consumer income levels, (t_1, t_2), such that there exists a 1:1 mapping from consumers into products. [**G-7.1**]

- If $c(u)$ is perfectly flat, than all consumers will choose to buy the highest quality product because the prices will all be the same. See diagrams in notes which show that for a certain flattness level of $c(u)$, even the consumer with the lowest income level, a, will choose the highest quality product, $\bar{u}$. [**G-7.2**] Thus, the mapping becomes degenerate and all consumer incomes levels in the open interval (t_1, t_2) are mapped directly into a single point, $\bar{u}$.

- So, in a world with a steep $c(u)$ mapping, there is room for a firm to enter in between two firms and gain market share while if $c(u)$ is flat, this breaks down. It can be shown that this is a necessary and sufficient condition for the finiteness property, ie, the "bound" in the example we did last week.

7.2 Part III: Markets with Incomplete Information

- This relaxes the 3rd assumption of competitive markets and Walrasian equilibrium. We begin with "Search Models" in which a consumer is searching for a low price in a market where different firms charge different prices.
- The simplest setting is one with a large number of firms and the distribution of prices among the firms is known. (Relaxing this assumption really doesn't affect the results.)
- We will study two types of models which are subclasses of the general model. 1) Sequential Search models and 2) Fixed Sample Size Searches.

7.2.1 The Sequential Search Model

- The consumer searches firms one at a time and each search costs $c > 0$.
- The Derivation. We aim to characterize a rule to follow in searching. Define the consumer's reservation price as follows:

$$R = Min_{Stopping\ Rules} \quad \Big\{ E(P_T + cT) \Big\}.$$

Here T is the number of time periods it takes to find a match. c is the cost of searching. P_T is the price paid at time T.

- Suppose a consumer has just been offered a price, P, where $P = R$. If he continues, he incurrs cost, R (assuming risk neutrality), and if he stops, he pays P. Label $P^* = R$.
- The form of the rule is as follows:

$$\text{Accept iff } P \leq P^* = R.$$

See handout for a diagram of the search process. Because the search process is recursive and the game looks the same at each stage, we can conclude:

$$R = P^* = c + E\{min(P, R)\}.$$

Hence we have that the reservation price for the consumer is equal to the cost of searching plus the minimum of the price he would pay if he stops or the reservation price he "loses" by continuing his search. We include the expectation because the value of P is unknown when the cost of c is incurred. (See diagram). **[G-7.3]**

- Plugging in $R = P^*$,

$$P^* = c + E\{min(P, P^*)\}.$$

Adding P^* outside the expectation and subtracting it from inside,

$$P^* = c + P^* + E\{min(P - P^*, P^* - P^*)\} = c + P^* + E\{min(P - P^*, 0)\}.$$
$$c = -E\{min(P - P^*, 0)\}.$$

- Rewriting,

$$c = \int_{P_{min}}^{P^*} (P^* - P) f(P) dP.$$

- And this is the relevant search rule. P^* is implicitly defined by this equation. See graph in notes, but the equation is intuitively obvious. **[G-7.4]** $P^* - P$ is the 1-period savings that you make by stopping. So the unit cost of searching, c, on the left, is equal to the expected value of the savings as P varies from P_{min} to P^*.

7.2.2 Remark (1) on the Search Models

- The first remark involves competition among firms. The reasoning goes that if setting one price is more profitable than setting another, all firms should make this move towards the optimal price and we would have a sort of price convergence.

- If we have a set of consumers who follow the sequential search rule and whose unit search costs are distributed on some interval,

$$0 < c_{min} \leq c \leq c_{max},$$

then there can be no non-degenerate distribution of prices which form a Nash Equilibrium. In other words for this distribution of search costs, all pricing distributions must collapse to a single price.

- To see this, consider any firm that sets price equal to P_{min}. We show that this cannot be an optimal reply. This follows from the fact that if this firm raises its price to any price in the interval,

$$P_{min} < P < P_{min} + C_{min},$$

then it has the same sales as before but strictly higher profits. To see why this firms sales remain the same, note that ALL consumers accept its price of P_{min}. A consumer faced with this price, could look elsewhere, but would incurr a cost of at least C_{min}. So as long as the firm doesn't raise its price by more than that minimum cost to the consumer, the consumer would prefer to buy from the firm at a price of slightly less than $P_{min} + C_{min}$ (which would at least be the price of the next best alternative product).

- So in this case, if we draw the demand schedule facing this firm, there is a vertical portion down around P_{min} such that the firm can raise its price by some amount, maintain the same sales level, and therefore increase profits. **[G-7.5]** Thus in this setting, this process will continue until prices all collapse to a single market price.

- Footnote. If $c_{min} = 0$, now there is NO vertical segment of the demand schedule and the above argument does not hold. BUT, if we calculate conditions for a NE in prices, it turns out that there exists a non-degenerate distribution of prices only if the distribution of search costs is rather extreme. $f(c)$ must be very high as $c \to 0$. This is a rather undesirable model of the distribution of search costs.

7.2.3 Remark (2) on the Search Models

- Forms of Search.

- The most popular alternative model is "Fixed Sample Size" search, or FSS. (Stigler). The idea is to choose the number of firms to search, say n, and pay $c \cdot n$ to get n quotes. Then choose the lowest. The optimal strategy is to choose n to minimize,

$$nc + E[P_{min}|n\ Searches].$$

- A hint on the class exercise on this model. Given a distribution $f(P)$, how can we find the expected value of P_{min}? Suppose $f(P) \sim U[0,1]$ as in the exercise. Therefore $f(P) = 1$ and $F(P) = P$. The probability that any one price is below P is $F(P) = P$. Thus,

$$\text{The Probability that any price is above } P = 1 - P.$$
$$\text{The Probability that all } n \text{ prices are above } P = (1-P)^n.$$
$$\text{The Probability that the minimum price is above } P = (1-P)^n.$$
$$\text{The Probability that the minimum price is below } P = 1-(1-P)^n.$$

- If there is no disadvantage of searching sequentially, it is better to do so, but if there are economies of scale in gathering quotes, then it is better to proceed by choosing n quotes for period 1, then examining P_{min}, and accepting or rejecting. If you continue, choose another n quotes in period 2 and so on ...

- The Sequential Search is the limiting case where the cost of doing n searches is equal to $c \cdot n$, or there are NO economies of scale in search.

- The FSS is a limiting case where there are strong economies of scale in search. Here the optimal value of n is such that the probability of getting a P_{min} which is accepted is high. Thus you are likely to stop after the first period.

- Footnote. One way of getting a model of market equilbrium with a dispersion of prices is to use FSS search.

- Next week we will look at a model by Salop and Stiglitz. Here a consumer is offered the opportunity to observe all prices for a fixed fee, c. The idea is that c is distributed on some interval. Consumers who have a low value of c will sample all prices and pay P_{min}. Consumers with a high value of c will choose a firm at random. Therefore we have in this case, an outcome with a 2-price equilibrium. Either set the "high price" or set price equal to P_{min}. In equilibrium, both strategies will be equally profitable.

8 Week 8: 4 Mar - 8 Mar

8.1 Salop Stiglitz Model of Search

- The idea: a consumer can buy a FULL set of quotes for a fixed cost, c. It follows that there will be 2 kinds of consumers: those with full information and those with no information.
- With this in mind, instead of starting with a continuous distribution of c, we will look at a simplified model where there are 2 groups of consumers, one with $c = 0$ (those that can afford the package of quotes) and those with a high value of c (those that must just search at random.)
- The model is as follows: Consider N consumers that each buy one unit of a product. There are a total of n firms in the market.
- A fraction, θ, are fully informed and thus the fraction $1 - \theta$ are uninformed.
- Consumers have a cut off price, p_0 above which, they will not buy. (A reservation price).
- Sales by a firm setting any price, p, such that $p_{min} < p \leq p_0$ are as follows:

$$q_m = \frac{1}{n}(1 - \theta)N.$$

  Note that firms that charge a price above the minimum price only get UNINFORMED consumers because the informed consumer will always buy at the minimum price. Note we have the fraction $\frac{1}{n}$ because these firms also compete for customers with firms that are charging p_{min}.

- The firms that are setting a price strictly above p_{min} will therefore raises their prices up to p_0 because by doing so, they raise profits without any loss in sales.
- Thus an equilibrium will be characterised by two prices, p_m and p_{min}.
- Sales by a firm setting p_{min} are as follows:

$$q_c = \underbrace{\frac{1}{n}(1 - \theta)N}_{Sales\ to\ Uninformed} + \underbrace{\frac{1}{K}\theta N}_{Sales\ to\ Informed}.$$

  Where K is the number of firms offering a price of p_{min}.

- Conditions for an Equilibrium.

– 1) A firm setting p_{min} must be on its marginal cost schedule, since otherwise, it can raises sales by an arbitrarily small price cut which will capture the entire market. Thus these firms compete as in a perfectly competitive market and hence the notation (q_c, p_c) to represent their sales and price will be used from now on. Thus,

$$MC(q_c) = p_c.$$

– 2) Equiprofit Condition: If either group was more profitable, one or more firms would switch groups. Thus, the notation for the quantity and price of the higher priced firm: q_m and p_m for the monopoly quantity and prices. Thus,

$$\pi_m = p_m q_m - TC(q_m) = \pi_c = MC(q_c) \cdot q_c - TC(q_c).$$

- See graph in notes for a pictures of the marginal cost function and the isoprofit curve. **[G-8.1]** We start at the price and quantity of the firms setting a price above p_{min} equal to p_m. Thus the combination (q_m, p_m) describes the one type of firm. Then to draw the isoprofit curve, it must slope downwards above the MC curve to maintain the same level of profits. Once it crosses the MC curve, the isoprofit curve slopes upwards.
- The first condition above says that (q_c, p_c) must be on the MC schedule and the second condition says that (q_c, p_c) must be on the isoprofit curve. Thus the intersection of the two determines the price and quantity of the low price firms.
- We can then consider the sales equation for the low priced firms:

$$q_c = \underbrace{\frac{1}{n}(1-\theta)N}_{Constant} + \underbrace{\frac{1}{K}\theta N}_{Depends\ -vely\ on\ K}.$$

  Thus as shown in the graph, we can take the quantity, q_c found above and plug it into this equation to solve for K, or the equilibrium number of low price firms.
- Thus, an equilibrium is characterised by 5 numbers: $(q_m, p_m), (q_c, p_c), K$.

8.2 Akerlof Model: The Market for Lemons

- The market involves imperfect information in regards to quality.
- The question is: Do bad products drive out good products?
- A general remark: Let a fraction, q, of NEW cars sold at average price, p, be lemons.
- Consider the market for 1-month old cars (these cars are old enough for the owner to know the quality of the car.)
- Since the average price of a new car is p, the value of a good car, v^g, must be greater than p. Similarly, the value of a bad car, v^b, must be less than p.

- Hence the expected value of a new car is:

$$E[v] = qv^b + (1-q)v^g.$$

- So now what price can trade take place? Well, the price of an old car must be the same no matter what the quality:

$$p^g = p^b = p^{old}.$$

- If $p^{old} \geq p$, the price of a new car, then owners of lemons will swap for new cars. Thus we assume $p^{old} < p$.
- Hence,

$$p^{old} < p \leq qv^b + (1-q)v^g = E[v].$$

Since v^g is greater than $E[v]$ and $E[v] \geq p > p^{old}$, this implies,

$$v^g > p^{old}.$$

- So the value of a good car to its owner is greater than the price he could obtain by selling it. Thus no good cars come to market. The owners of good cars are said to be "Locked In."
- Consider the following example of "Market Breakdown." Let x be a quality index. Let two groups of traders have utility function, income, and ownership as follows:

$$U_1 = M + \sum_{i=1}^{N} x_i, \qquad Income = Y_1, \qquad Own\ N\ Cars.$$

$$U_2 = M + \sum_{i=1}^{N} \frac{3}{2} x_i, \qquad Income = Y_2, \qquad Own\ 0\ Cars.$$

So the type 1 traders can be thought of as the Dealers and the type 2 traders are the consumers. M is the hicksian composite commodity.

- Under full information, dealers sell cars to consumers because $\frac{3}{2} > 1$.
- Let p be the price and μ be the average quality of traded cars.
- Consider type 1 traders: the Dealers. They only have information about the quality of their own cars so they will buy a car at price p so long as its expected quality, $E(x) = \mu > p$. Thus their demand schedule is as follows:

$$D_1 = \begin{cases} \dfrac{Y_1}{p} & \text{if } \dfrac{p}{\mu} \leq 1. \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

- Similarly the demand by type 2 traders, the consumers, is as follows:

$$D_2 = \begin{cases} \dfrac{Y_2}{p} & \text{if } \dfrac{p}{\mu} \leq \dfrac{3}{2}. \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

- Adding the demand schedules together yields:

$$D(p, \mu) = \begin{cases} \dfrac{Y_1 + Y_2}{p} & \text{if } \dfrac{p}{\mu} \leq 1. \\ \\ \dfrac{Y_2}{p} & \text{if } 1 < \dfrac{p}{\mu} \leq \dfrac{3}{2}. \\ \\ 0 & \text{if } \dfrac{p}{\mu} > \dfrac{3}{2}. \end{cases} \tag{8}$$

- Supply. The only source of supply is from the type 1 traders. Let the quality, x be distributed uniformly on the interval $[0, 2]$. So if all cars come to market,

$$\text{Average Quality} = E[x] = \int_0^2 x \frac{1}{2} dx = 1.$$

- Note a type 1 trader will sell a car of quality x at any price $p \geq x$. So for any price, p, the cars supplied are those with $x \leq p$. **[G-8.2]** Thus given that $x \sim U[0, 2]$ and $f(x) = \dfrac{1}{2}$, the fraction of cars supplied at any price p is $\dfrac{p}{2}$. Thus given there are N cars in the market, the total supply of cars is:

$$S_1 = \frac{p}{2} N.$$

- Since the cars supplied to market have quality ranging from 0 up to p, and the distribution is uniform, the average quality of those cars that come to market is $\frac{p}{2}$. Thus,

$$\mu = \frac{p}{2}.$$

Or,

$$\frac{p}{\mu} = 2.$$

See graph in notes which show the demand function and the supply function that we just derived. **[G-8.3]** Clearly since the price of cars brought to market is a constant $\frac{p}{\mu} = 2$, and the willingness to pay (even for the first unit) is only equal to $\frac{p}{\mu} = \frac{3}{2}$, we have "market breakdown".

8.3 Spence Model: Market Signalling

- Here we have two groups of possible employees that a firm is considering hiring. Group I has marginal product equal to 1, they are a proportion of the population equal to q_1 and their cost of attaining education level y is equal to y. Group II has marginal product equal to 2, they are a proportion of the population equal to q_2 and their cost of attaining education level y is equal to $\frac{1}{2}y$.
- The signal to the employer is investment in education. Notice that the more productive of the two groups can make this investment more efficiently.
- We seek a "Signalling Equilibrium." We will look for critical level of education, y^*, such that workers are paid their HIGH level of marginal product (2) if their educational level is above y^* and workers are paid their lower level of marginal product (1) if their education is below y^*. In equilibrium, workers will sort themselves by their choice of y. So at equilibrium (called a seperating equilibrium), workers are correctly identified.
- See graph in notes which shows that the lower productivity group will maximize their cost of education over their expected returns and thus gain zero education. **[G-8.4]** Similarly, the high productivity group will obtain education exactly equal to y^*. Thus the educational cutoff level, y^*, successfully seperates the two groups and provides the employer with this valuable information.
- Notes.
  - 1) Individually this is rational, but it is NOT socially optimal.
  - 2) Beliefs of the employer (his choice of y^*) conditions the outcome. So if y^* is set incorrectly, the signalling breaks down.
  - 3) Group I (the unproductive type), are worse off then under no-signalling (ns) where everyone receives the same wage:

$$w_{ns} = q_1(1) + (1 - q_1)(2) = 2 - q_1 = \text{Average Marginal Product.}$$

  - 4) Group II may be worse off depending on the parameter values in the model. Under signalling (s), group II gets:

$$w_s = 2 - C(y^*).$$

  Under no-signalling (ns):

$$w_{ns} = 2 - q_1.$$

Thus, $w_{ns} > w_s$ if:

$$2 - q_1 > 2 - C(y^*).$$

$$q_1 < C(y^*).$$

For example if $q_1 = \frac{1}{2}$ and $y^* > 1$. Thus $C(y^*) > \frac{1}{2}$. Thus $C(y^*) > q_1$ which means group II would prefer no-signalling.

8.4 Part IV of Lent Term: General Equilibrium, Trade and Welfare

- See handout for derivations. We have a 2-sector model, Manufactures and Food (m and f), and 2 factors of production, Labor (L) and Land (T).
- Assume the production function exhibits constant returns to scale. This along with perfect competition ensures that factor payments exhaust revenue.
- Prices will be driven down to their unit cost of production.
- The Analysis proceeds in two steps. 1.) We look at at small open economy facing fixed world prices for goods. Here both factor endowments, L and T, and prices P_m, and P_f are taken as parameters. 2.) Solve.
- More next week.

9 Week 9: 11 Mar - 15 Mar

9.1 More on the General Equilibrium Model

9.1.1 The Basic Trade Theorems

- The only relavent thing to add beyond what is on the sheet: We assume that the manufacturing sector is relatively LABOR intensive. This will set up the upcoming theorems.

- The Three Basic Trade Theorems.

- The Factor Price Equilization.

  - Factor prices determine commodity prices. ie, if we change factor endowments, holding world commodity prices constant, this does not affect domestic factor prices.
  - We should not take this theorem too seriously because it depends heavily on two assumptions: 1.) Production functions are the same everywhere and 2.) We have an internal solution (normally we see corner solutions in reality as countries often produce 0 units of one good and trade something else they are good at making to get units of the good that they do not make.
  - Thus, this theorem will only apply to countries that are relatively similar in terms of factor endowments and technology.

- The Rybczynski Theorem. [**G-9.1**]

  - Let the endowment of one factor (say labor, L) rise. Thus $L^* > 0$ and $T^* = 0$. Holding p_m and p_f constant, or

    $$p_m^* = p_f^* = w^* = r^* = 0.$$

    Thus $M^* > L^*$ and $F^* < 0$.
  - Thus, in words, if one factor endowment (say labor, L) rises, then the output of the labor intensive commodity rises and the output of the other commodity falls.
  - More generally, For any L^* and T^* such that $L^* > T^*$, then,

    $$M^* > L^* > T^* > F^*.$$

- The Stolper - Samuelson Theorem.

  - Keep factors, Labor and Land (L and T) fixed but raise p_m say by a tariff on manufacturers. Let p_f remain fixed. The theorem says this raises the return, w, to labor, that is, the factor used intensively in manufacturers and lowers the return to capital, r. Thus, $w^* > p_m^*$ and $r^* < 0$.

- More generally, if $p_m^* > p_f^*$, then,

$$w^* > p_m^* > p_f^* > r^*.$$

- The key assumption underlying this theorem is that is only applies in the long run. Factors must be able to be moved from one industry to another which usually takes a long time.
- In the short run, Mcgee observed that when a labor intensive industry looks for tariff protection, this demand is normally supported by both labor and management in that industry. This goes against the theorem, but again, is only true in the short run so the theorem does not apply.

- See graph in notes [**G-9.1**] for a graphical interpretation of the Rybczynski theorem. We see that when labor is expanded and we have assumed homothetic production isoquants, the output of the land intensive commodity falls while the output of the labor intensive commodity rises.

- Under the "Closing the model" section of the handout, there are only a few equations that are actually important. Equation (b):

$$w^* - r^* = \frac{1}{|\theta|}(p_m^* - p_f^*).$$

Equation S, (Supply):

$$M^* - F^* = \frac{1}{|\lambda|}(L^* - T^*) + \sigma_S(p_m^* - p_f^*).$$

This equation summarizes the supply side. We also have the equation D, (Demand):

$$M^* - F^* = -\sigma_D(p_m^* - p_f^*).$$

- Combining Equation S and Equation D:

$$p_m^* - p_f^* = -\frac{1}{|\lambda|(\sigma_S + \sigma_D)}(L^* - T^*).$$
$$M^* - F^* = \frac{\sigma_D}{|\lambda|(\sigma_S + \sigma_D)}(L^* - T^*).$$

This specifies the effect of a change in endowments on output and prices.

- The point to note is: The impact on commodity outputs is less if either 1) σ_S is larger, or 2) σ_D is smaller. Note in the $M^* - F^*$ equation, we have the term:

$$\frac{\sigma_D}{(\sigma_S + \sigma_D)} = \frac{1}{1 + (\sigma_S/\sigma_D)}.$$

So if σ_D is larger, or the elasticity of substitution between commodities on the demand side is larger, $M^* - F^*$ is larger which means the impact on commodity outputs is greater.

10 Week 10: 18 Mar - 22 Mar

10.1 General Equilibrium - Taxes and Subsidies

- We would like to study the incidence of taxes and subsidies. Who bears the burden of each? Since a tax and a subsidy have a symmetric, though opposite, influence, we'll only study an example of a subsidy.
- See handout for details. But we see that through the introduction of a subsidy, we can derive equations for the effects on commodity prices, factor prices, and finally, commodity prices.
- Consider an example. Suppose we impose a 10 percent subsidy on the producers of good M (which is labor intensive).
- The result is an increase in the return to the factor used intensively in M (labor).
- The price of M falls.
- The relative effects of each of these depends on the elasticity of substitution on the demand and supply sides.
- We show that if $\sigma_D \longrightarrow 0$ (goods complementary), the subsidy is entirely passed on to the consumer in form a price fall and the factor prices and commodity endowments do not change. If $\sigma_D \longrightarrow \infty$ (substitutable goods), the price change is zero, but the return on labor increases by a proportion of the subsidy.
- We have similar results when looking at σ_S. See handout.

10.2 Trade Theory

- Motivation: Consider the notion of inter-industry trade versus intra-industry trade. Inter-industry trade occurs between two countries which trade the goods they produce most efficiently for goods that they are less efficient in production. Intra-Industry trade consists of two countries trading goods in the same industry but which are slightly differentiated.
- The typical model is the hotelling circular road model where two countries come together and the firms are now placed equally around a circle of twice the size (as was done in a problem set).
- The key idea is that we combine two assumptions: 1) Products are differentiated and 2) Products are produced under increasing returns to scale (or falling average cost curves). Note that with CRS, or a cost schedule of the form:

$$C(X) = \beta X,$$

  where $MC = \beta$ and fixed costs $= 0$, then free entry would induce an infinite number of firms around the circle.

- Thus we need to introduce a setup cost, α, to avoid this problem. Assume a cost function of the form:

$$C(X) = \alpha + \beta X.$$

Where α is a fixed setup cost and β is the marginal cost of production.

- In order to motivate the next idea, we will first return to the problem we developed in the problem set:

  - We joined two economies under the circular road hotelling model and found that prices fell from $MC + \frac{t}{4}$ to $MC + \frac{t}{8}$.
  - In the long run, note that with $2N = 8$ firms present, profits are lower so if profits were just sufficient to cover fixed outlays or setup costs ex-ante, now profits will be too low to cover fixed costs.
  - We distinguish two cases: 1) This fixed outlay, α, is fixed but NOT SUNK. Then we get immediate exit. Or 2), α is a SUNK cost. Then this adjustment takes longer as when plant equipment needs replacement, it will no longer pay to replace it, so we'll get a reduction in the number of plants over time.
  - How many plants will we have in the long run? Well, we could figure it out precisely using a zero profit condition but we can say for sure that if there were N firms before and a potential of $2N$ firms afterwards, then the number of long run firms, n, must be:

    $$N < n < 2N.$$

    This can be seen by showing that having either N or $2N$ firms is not an equilibrium. If there are N firms in the market after the merger, then prices must rise again to their old level, but sales per firm has doubled. Thus this induces higher profits and entry. If there are $2N$ firms in the market after the merger, this means that price is lower but sales per firm are the same (as the number of consumers has also doubled). Thus this induces lower profits and exit.

- So given this example, we get the next idea: the impact of trade involves two channels. First prices are lower and second, the number of varieties available to a consumer is strictly higher. So the focus of our interest in the next model will be on the relative sizes of these two effects. We will show that the relative size depends on two factors:

  - 1) The degree of increasing returns, as measured by α.
  - 2) The intensity of consumer tastes, as measured by t.

10.2.1 The Krugman Model

- In this model we try to represent consumer preferences using an alternative model of horizontal product differentiation.

- The idea is that all consumer have the same utility function characterised by a taste for variety:

$$U = \sum_i v(c_i).$$

With $v' > 0$ and $v'' < 0$ as usual. c_i denotes the consumption of variety i.

- We immediately introduce a technical assumption:

$$\epsilon(c_i) = \frac{-v'}{v'' c_i} \text{ is decreasing.}$$

Note that we will show that ϵ represents the elasticity of demand facing a firm which would always be decreasing for a linear demand but we have the assumption that it is always decreasing for any demand schedule.

- We set up a simple general equilibrium model as follows.

- Production. Suppose there is only ONE factor of production, labor, call it L. Thus the amount of labor, l_i used to produce output, X_i, of good i is:

$$l_i = \alpha + \beta X_i.$$

- Total production of good i is denoted X_i. The number of individuals in the population is L. In a symmetric equilibrium, the consumption of each good, c_i, is the same and:

$$X_i = Lc_i.$$

Where c_i is the per capita consumption now since $c_1 = c_2 = c_3 = \dots$.

- Full Employment. We can write labor supply equals labor demand as follows:

$$\underbrace{L}_{\textit{Labor Supply}} = \underbrace{\sum_i l_i}_{\textit{Labor Demand}} = \sum_i (\alpha + \beta X_i).$$

- WE SEEK a symmetric NE with $P_i = P$ as the price of good i and $X_i = X$ for all i.

- We proceed in 3 steps.

- Step I: Determining Demand.

- Analyze the demand schedule facing the firm. The individual chooses c_i to maximize his utility. Even without the actual form of the utility function, we know that the FOC implies:

$$\frac{MU_1}{P_1} = \frac{MU_2}{P_2} = \frac{MU_3}{P_3} = \ldots.$$

  So, written another way:

$$v'(c_i) = \lambda p_i \ \ \forall\, i.$$

  Where λ is the lagrange multiplier.

- Since $X_i = Lc_i$, substituting in,

$$v'(X_i/L) = \lambda p_i.$$

  Or,

$$p_i = \lambda^{-1} v'(X_i/L).$$

  And this is our demand schedule (DD).

- Now as an aside, as can show that $\epsilon(c_i)$ defined earlier is the elasticity of demand. Given the demand schedule written above, we have:

$$\frac{dp_i}{dX_i} = \lambda^{-1} v''(X_i/L)\frac{1}{L}.$$

  Thus, the elasticity of demand is:

$$-\frac{P_i}{X_i}\frac{dX_i}{dP_i} = \frac{-\lambda^{-1}v'}{X_i}\frac{1}{\lambda^{-1}v''\frac{1}{L}} = -\frac{v'}{v''\frac{X_i}{L}} = -\frac{v'}{v''c_i} = \epsilon(c_i).$$

• Step II: Determining Price.

- We focus immediately on the case with a large number of firms. Here the change in a single price has a negligable effect on λ. So we approximate by treating λ as constant. Thus,

$$\pi_i = P_i X_i - (\alpha + \beta X_i)w.$$

  Where w is the wage rate.

- Thus, noting that $X_i = X_i(P_i)$:

$$\frac{d\pi}{dP_i} \Longrightarrow X_i + P_i\frac{dX_i}{dP_i} - \beta w\frac{dX_i}{dP_i} = 0.$$

  Multiply through by $\frac{dP_i}{dX_i}$,

$$X_i \frac{dP_i}{dX_i} + P_i - \beta w = 0.$$

Dividing through by P_i,

$$\frac{X_i}{P_i}\frac{dP_i}{dX_i} + 1 - \frac{\beta w}{P_i} = 0.$$

Or,

$$-\frac{1}{\epsilon} + 1 = \frac{\beta w}{P_i}.$$

$$\frac{\epsilon - 1}{\epsilon} = \frac{\beta w}{P_i}.$$

Thus,

$$P_i = \frac{\epsilon}{\epsilon - 1}\beta w.$$

Or,

$$\frac{P_i}{w} = \frac{\epsilon}{\epsilon - 1}\beta.$$

So this is our "Pricing Rule."

- Step III: Free Entry Condition.
  - We now fix the number of firms (or varieties) by appealing to free entry with a large number of firms and symmetry, this reduces to a zero profit condition such that:

$$\pi_i = P_i X_i - (\alpha + \beta X_i) w = 0.$$

Or,

$$P_i X_i = (\alpha + \beta X_i) w.$$

$$\frac{P_i}{w} = \frac{(\alpha + \beta X_i)}{X_i}.$$

$$\frac{P_i}{w} = \frac{\alpha}{X_i} + \beta.$$

Since $X_i = Lc_i$,

$$\frac{P_i}{w} = \frac{\alpha}{Lc_i} + \beta.$$

And this is our "Free Entry Condition."

- See graph in notes which shows the Pricing Rule and the Free Entry Condition. [**G-10.1**] This characterises $\frac{P}{w}$ and c. To find a number of firms (or varieties), n, we know:

$$L = (\alpha + \beta X)n.$$

So,

$$n = \frac{L}{(\alpha + \beta X)}.$$

- To analyse the trade, we join two identical economies. [**G-10.2**] This is equivalent to doubling L. See graph in notes which shows shift of free entry condition and the result is a lower equilibrium price and a lower level of consumption.

- The first effect: $\frac{P}{w}$ falls. The interpretation is best seen looking at a "Chamberlinian" equilibrium where the demand schedule is just tangent to the average cost schedule and the firm produces exactly at this tangency were profits are zero. [**G-10.3**] The rise in L raises X, the output per firm, thus average cost falls supporting a price fall.

- The second effect: c falls. Since:

$$\frac{P}{w} = \beta + \frac{\alpha}{Lc} = \beta + \frac{\alpha}{X}.$$

It follows that output per firm, $X = \frac{\alpha}{(P/w) - \beta}$ rises.

- Finally the number of firms (or varieties),

$$n = \frac{L}{\alpha + \beta X} = \frac{L}{\alpha + \beta Lc},$$

rises since c falls.