BOARD OF GOVERNORS OF THE FEDERAL RESERVE SYSTEM
DIVISION OF RESEARCH AND STATISTICS

Date: October 6, 2003

To: Distribution

From: Hao Zhou and Matthew Chesnes

Subject: VIX Index Becomes Model Free and Based on S&P 500

1 Introduction

On September 22, 2003, the Chicago Board Options Exchange made some major changes to the way the implied volatility index (VIX) is constructed. It is now based on the more liquid S&P 500 index options, instead of the S&P 100. Even more important is a change in the methodology for constructing the index. The old index was a weighted average of implied volatilities inverted from the Black-Scholes option pricing model. The new index eliminates this dependence on a specific model and uses a model-free approach similar to the one proposed in Britten-Jones and Neuberger (2000).

2 Old Methodology

The old VIX index is based on the Black-Scholes implied volatility of S&P 100 options. To construct the old VIX, two puts and two calls for strikes immediately above and below the current index are chosen. Near maturities (greater than eight days) and second nearby maturities are chosen to achieve a complete set of eight options. By inverting the Black-Scholes pricing formula using current market prices, an implied volatility is found for each of the eight options. These volatilities are then averaged, first the puts and the calls, then the high and low strikes. Finally, an interpolation between maturities is done to compute a 30 calendar day (22 trading day) implied volatility. (See appendix for details.)[1]

Because the Black-Scholes model assumes the index follows a geometric Brownian motion with constant volatility, when in fact it does not, the old VIX will only approximate the true risk-neutral implied volatility over the coming month. In reality the price process is likely more complicated than geometric Brownian motion. Limiting it to a very specific form and deducing an implied volatility from market prices may lead to substantial error in the estimation.

[1] Since the S&P 100 index options are American, an approximation is involved to compute the implied volatility.

3 New Methodology

Britten-Jones and Neuberger (2000) develop a model-free risk-neutral implied volatility over a future time period

$$E_0^Q\left[\int_0^T \left(\frac{dS_t}{S_t}\right)^2\right] = 2\int_0^\infty \frac{C(T, Ke^{rT}) - \max(S_0 - K, 0)}{K^2} dK, \tag{1}$$

which is completely specified by the set of options expiring at that date. The price and volatility processes are NOT assumed to follow a specific model, but only required to satisfy the following assumptions: (1) Markovian, (2) continuity, and (3) no-arbitrage.[2]

The new S&P 500 implied volatility index is implemented by CBOE using the following formula

$$\sigma^2 = \frac{2}{T}\sum_i \frac{\Delta K_i}{K_i^2} e^{rT} Q(K_i) - \frac{1}{T}\left[\frac{F}{K_0} - 1\right]^2. \tag{2}$$

One of the advantages of this approach is that all available out-of-the money call and put options are utilized instead of just the eight used in the old VIX. $Q(K_i)$ is the midprice of the bid/ask spread for the option with strike K_i. The last term in equation (2) is intended to adjust for the fact that there is no exact at-the-money option.[3] ΔK_i is the interval between strikes on either side of K_i. And T is the time to maturity which is now based in minutes instead of days as in the old VIX. So formula (2) can be viewed as a crude approximation of equation (1).[4]

Aside from incorporating a wider range of strikes, the main virtue of the new VIX index is that the price process is not assumed to follow a geometric Brownian motion or any other specific model. The new VIX is therefore a much better model-free estimate of the true expectation of volatility over the coming month.

[2]Notice the right hand side of equation (1) involves an integral over a continuum of strike prices. Since strike prices on the options market are only available in discrete intervals, Jiang and Tian (2003) have developed an accurate discretization procedure to deal with this issue. (See appendix for discrete formula).

[3]F is the forward price defined as:

$$F = K^* + e^{rT}(C^* - P^*),$$

where K^*, C^*, and P^* are the at-the-money strike, call and put prices. K_0 is the strike immediately below the forward price, F.

[4]To complete the calculation of the new VIX, σ^2 is calculated for options maturing at the next two expiration dates (of at least eight days). Then one can interpolate (or occasionally extrapolate) to get a 30 calendar day implied volatility.

$$VIX_{new} = 100\sqrt{\left\{T_1\sigma_1^2\left[\frac{N_{T_2} - N_{30}}{N_{T_2} - N_{T_1}}\right] + T_2\sigma_2^2\left[\frac{N_{30} - N_{T_1}}{N_{T_2} - N_{T_1}}\right]\right\}\frac{N_{365}}{N_{30}}}.$$

N_{T_1} and N_{T_2} are the number of minutes until maturities T_1 and T_2 respectively.

4 Comparing the New and Old Indices

Since the new index has been created back through 1990, we can compare the historical performance of the two indices. As seen in figure 1, the indices follow each other fairly closely. Most of the deviation occurs between 1996 and 1997 and then again in recent years.

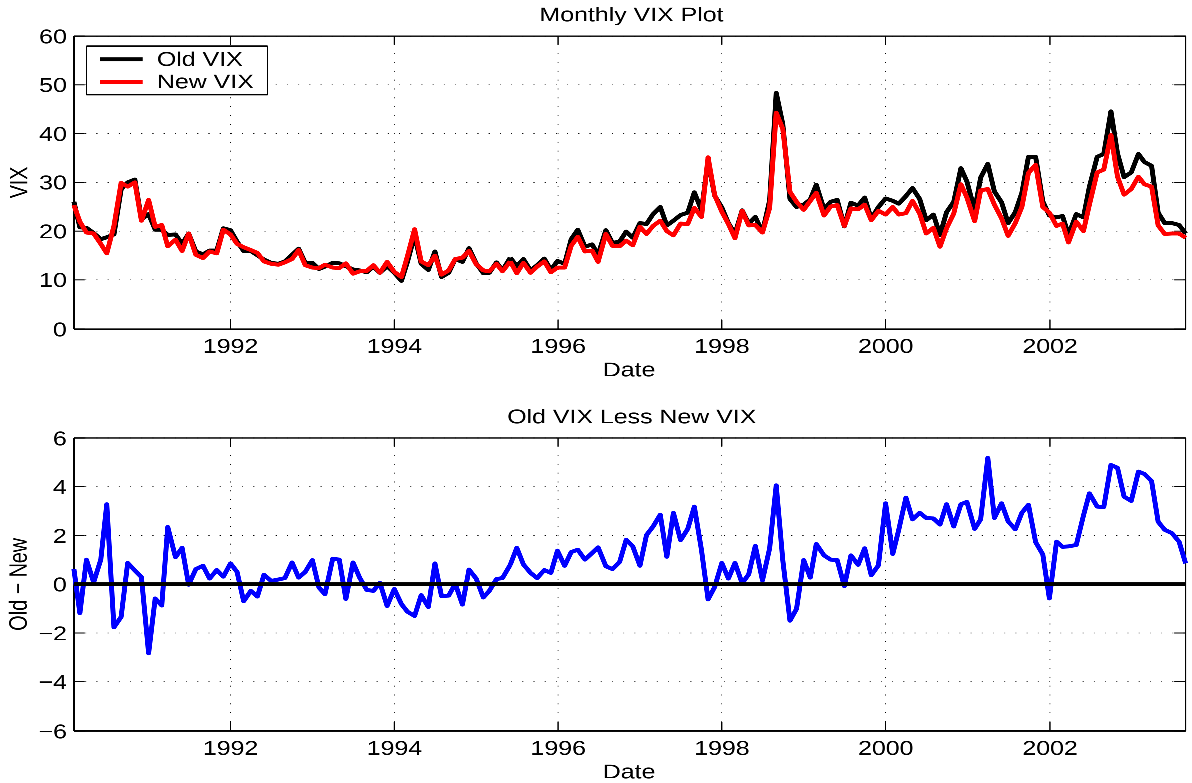

Figure 1: Monthly Time Series and Difference.

Table 1 displays descriptive statistics on the two indices. They are generated from end of the month observations from January, 1990 through August, 2003. The average and spread of the index over time is slightly smaller for the new index, but both measures are very persistent.

5 Estimating Volatility Risk Premium

One of the advantages of having the model-free implied volatility available is that we can estimate the volatility risk premium based on a more solid ground (see Bollersev, Gibson, and Zhou 2003 for a discussion of the methodology). Suppose that log price ($p_t = \log S_t$) and stochastic volatility

Table 1: Descriptive Statistics on New and Old Indices

Statistic	Old VIX	New VIX
Mean	21.386	20.241
Standard Deviation	7.306	6.498
Skewness	0.765	0.779
Kurtosis	3.654	3.720
AR(1)	0.849	0.831

are following the processes

$$\begin{aligned} dp_t &= \mu_t dt + \sqrt{V_t} dB_t, \\ dV_t &= \kappa(\theta - V_t)dt + \sigma\sqrt{V_t} dW_t, \\ \text{corr}(dB_t, dW_t) &= \rho, \end{aligned} \tag{3}$$

where the instantaneous correlation between the two Brownian motions drives the so-called leverage effect. Under a no-arbitrage argument, there exists a risk-neutral distribution

$$\begin{aligned} dp_t &= r_t^* dt + \sqrt{V_t} dB_t^*, \\ dV_t &= \kappa^*(\theta^* - V_t)dt + \sigma\sqrt{V_t} dW_t^*, \\ \text{corr}(dB_t^*, dW_t^*) &= \rho, \end{aligned} \tag{4}$$

where r_t^* denotes the risk-free interest rate. The values of the risk-neutral parameters in (4) are directly related to the parameters of the actual price process in equation (3) by the functional relationships, $\kappa^* = \kappa + \lambda$ and $\theta^* = \kappa\theta/(\kappa + \lambda)$. With the availability of model-free implied volatility and realized volatility, and using the estimation strategy developed in Bollersev, Gibson, and Zhou (2003), we have the following parameter estimates (with standard errors):

Table 2: Estimation of Underlying Model and Risk Premium

Parameter	Old VIX Estimate	New VIX Estimate
κ	0.1160 (0.0354)	0.1187 (0.0695)
θ	12.6516 (0.4022)	11.9057 (1.0691)
σ	1.6953 (0.5633)	1.5521 (0.5499)
λ	-1.9485 (0.0436)	-1.8478 (0.0603)
ρ	-0.5458 (0.2741)	-0.4673 (0.2444)

It is clear from Table 2 that using the old VIX index, based on the Black-Scholes implied volatility, results in qualitatively similar estimates, compared with the new VIX index, based on the model-free approach. However, quantitatively, the old VIX index implies a more volatile process (θ is 5% higher and σ is 10% higher), a higher risk premium (λ is 5% more negative), and an exaggerated leverage (ρ is 20% more negative).

References

Bollerslev, Tim, Michael Gibson, and Hao Zhou, 2003, “Estimating Risk-Neutral and Objective Dynamics Using Model-Free Implied and Realized Volatilities,” *Work in Progress*.

Britten-Jones, Mark and Anthony Neuberger, 2000, “Option Prices, Implied Price Processes, and Stochastic Volatility,” *Journal of Finance*, 55, 839-866.

Jiang, George J and Yisong S. Tian. “Model-Free Implied Volatility and Its Information Content.” *Working Paper.* March, 2003.

Whaley, Robert E. “The Investor Fear Gauge.” *Journal of Portfolio Management.* 26 (2000), 12-17.

Appendix

Old VIX Construction

Following the notation in Whaley (2000), denote the strike prices below and above the current index level (S) as X_l and X_u respectively. The implied volatilities used to create the VIX index are as follows:

Exercise	Nearby Contract(1)		Second Nearby Contract(2)	
price	Call	Put	Call	Put
$X_l(< S)$	$\sigma_{c,1}^{X_l}$	$\sigma_{p,1}^{X_l}$	$\sigma_{c,2}^{X_l}$	$\sigma_{p,2}^{X_l}$
$X_u(> S)$	$\sigma_{c,1}^{X_u}$	$\sigma_{p,1}^{X_u}$	$\sigma_{c,2}^{X_u}$	$\sigma_{p,2}^{X_u}$

The first step is to average the put and call implied volatilities for each strike and maturity to reduce the number of volatilities to 4. Compute:

$$\sigma_1^{X_l} = (\sigma_{c,1}^{X_l} + \sigma_{p,1}^{X_l})/2, \qquad \sigma_1^{X_u} = (\sigma_{c,1}^{X_u} + \sigma_{p,1}^{X_u})/2,$$
$$\sigma_2^{X_l} = (\sigma_{c,2}^{X_l} + \sigma_{p,2}^{X_l})/2, \quad \text{and} \quad \sigma_2^{X_u} = (\sigma_{c,2}^{X_u} + \sigma_{p,2}^{X_u})/2.$$

Now average the implied volatilities above and below the index level as follows:

$$\sigma_1 = \sigma_1^{X_l}\left(\frac{X_u - S}{X_u - X_l}\right) + \sigma_1^{X_u}\left(\frac{S - X_l}{X_u - X_l}\right) \quad \text{and} \quad \sigma_2 = \sigma_2^{X_l}\left(\frac{X_u - S}{X_u - X_l}\right) + \sigma_2^{X_u}\left(\frac{S - X_l}{X_u - X_l}\right).$$

The final step in calculating the VIX is to interpolate between (or extrapolate from) the two maturities to create a 30 calendar day (22 trading day) implied volatility index.

$$VIX_{old} = \sigma_1\left(\frac{N_{t_2} - 22}{N_{t_2} - N_{t_1}}\right) + \sigma_2\left(\frac{22 - N_{t_2}}{N_{t_2} - N_{t_1}}\right),$$

where N_{t_1} and N_{t_2} are the number of trading days to maturity of the two contracts.

Discretization Formula by Jiang and Tian (2002)

The discrete form of the model-free risk-neutral implied volatility is:

$$E_0^Q\left[\sum_{i=0}^{n-1}\left(\frac{S_{t_{i+1}} - S_{t_i}}{S_{t_i}}\right)^2\right] = \left(u - \frac{1}{u}\right)\sum_{i=-m}^{m}\frac{C(T, K_i e^{rT}) - \max(S_0 - K_i, 0)}{K_i},$$

where

$$\begin{aligned} t_i &= ih, \text{ for } i = 0, 1, 2, \ldots, n, \\ h &= \frac{T}{n}, \\ K_i &= S_0 u^i, \text{ for } i = 0, \pm 1, \pm 2, \ldots, \pm m, \\ u &= (1 + k)^{\frac{1}{m}}. \end{aligned}$$

This method yields approximation errors depending on the size of the grid chosen, though Jiang and Tian find that these errors are negligible when $k \geq 0.2$ and $m \geq 20$.